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Nonequilibrium Quantum Dynamics of Second Order Phase Transitions

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Abstract

We use the so-called Liouville-von Neumann (LvN) approach to study the nonequilibrium quantum dynamics of time-dependent second order phase transitions. The LvN approach is a canonical method that unifies the functional Schrödinger equation for the quantum evolution of pure states and the LvN equation for the quantum description of mixed states of either equilibrium or nonequilibrium. As nonequilibrium quantum mechanical systems we study a time-dependent harmonic and an anharmonic oscillator and find the exact Fock space and density operator for the harmonic oscillator and the nonperturbative Gaussian Fock space and density operator for the anharmonic oscillator. The density matrix and the coherent, thermal and coherent-thermal states are found in terms of their classical solutions, for which the effective Hamiltonians and equations of motion are derived. The LvN approach is further extended to quantum fields undergoing time-dependent second order phase transitions. We study an exactly solvable model with a finite smooth quench and find the two-point correlation functions. Due to the

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spinodal instability of long wavelength modes the two-point correlation functions lead to the $t^{1/4}$-scaling relation during the quench and the Cahn-Allen scaling relation $t^{1/2}$ after the completion of quench. Further, after the finite quench the domain formation shows a time-lag behavior at the cubic power of quench period. Finally we study the time-dependent phase transition of a self-interacting scalar field.

I. INTRODUCTION

A system can interact directly with an environment to make its coupling parameters depend explicitly on time. Even the effective coupling parameters of a subsystem of a closed system, though conserved as a whole, may depend implicitly on time through an interaction with the rest of the system. The characteristic feature of these open systems is that their effective coupling parameters depend explicitly or implicitly on time. Therefore the genuine understanding of these systems requires the real-time processes from their initial conditions. Of a particular interest are the systems undergoing phase transitions. When a system cools down through an interaction with an environment, it may undergo a phase transition and its coupling parameters depend explicitly on time. Similarly, matter fields in the expanding early Universe undergo phase transitions through the interaction with gravity.

Phase transitions are one of the most physically important phenomena in nature and have wide applications from condensed matter physics, particle physics and even to cosmology. The Kibble mechanism explains formation of topological defects in symmetry breaking phase transitions [1]. The kinetic process by Zurek has revealed a new feature of symmetry breaking phase transitions [2]. The dynamics of phase transitions and formation of topological defects since then have become an important tool in variety of phenomena in condensed matter physics [3]. It is also widely accepted that symmetry breaking phase transitions in the early Universe are inevitable for the structure formation of the present Universe [4]. There has been an attempt through laboratory experiments to investigate the process of structure formation in the early stage of Universe [5]. In QCD the quark-antiquark condensate breaks chiral symmetry when the temperature of quark-gluon plasma is lowered [6,7].

However, the most difficult and subtle facet of phase transitions is to understand its dynamics and the formation process of topological defects. As emphasized above, systems do become out of equilibrium in general during phase transitions because their coupling parameters depend explicitly on time through the interaction with an environment (heat bath). Hence the nonequilibrium dynamics of phase transitions should differ significantly from the equilibrium dynamics. In finite temperature field theory one obtains the effective action for the system in a thermal equilibrium or quasi-equilibrium by calculating its quantum fluctuations about a vacuum [8]. But as the phase transition proceeds, the fluctuations grow and the stability is lost. The effective action, when extrapolated to the phase transition regime in a literal sense, has a complex value, the imaginary part of which is related with the decay rate of the false vacuum [9]. In this sense finite temperature field theory can not be directly applied to study symmetry breaking phase transitions.

Schwinger and Keldysh introduced the closed time-path integral to treat properly the quantum evolution of quantum fields out of equilibrium from their initial thermal equilibrium [10]. Since then the closed time-path integral method has been developed and applied to many related problems [11]. Recently the closed time-path integral method has been employed to study the nonequilibrium dynamics of second order phase transitions [12–16]. Another method is the functional Schrödinger-picture approach, in which the evolution of quantum states is found for explicitly time-dependent Hamiltonians [17]. The large N-expansion method [18,19] and the mean-field or Hartree-Fock method [20–22] are used in conjunction with either the Schwinger-Keldish or functional Schrödinger method. Still another methods are the time-dependent variational principle [23], the generating function for

correlation functions [24], and thermal field theory [25].

The purpose of this paper is two-fold. In the first part of this paper we elaborate and establish the recently introduced Liouville-von Neumann (LvN) approach so that it can readily be applied to nonequilibrium dynamics. In the second part we apply the approach to the systems undergoing time-dependent second order phase transitions. The LvN approach is a canonical method that unifies the functional Schrödinger equation for the quantum evolution of pure states and the LvN equation for the quantum description of mixed states of either equilibrium or nonequilibrium. One of the advantages is that one can make use of the well-known techniques of quantum mechanics and quantum many-particle systems. It is based on the observation by Lewis and Riesenfeld [26] that the quantum LvN equation, which is originally used to define the density operator for a mixed state, can also be used to find the exact pure states of a time-dependent harmonic oscillator. This observation makes it possible to find not only the mixed state but also the pure state of a time-dependent system. This LvN approach has been developed and applied to quantum fields in an expanding Friedmann-Robertson-Walker universe [27–29] and to open boson and fermion systems [30].

In this paper we particularly focus on the model systems whose coupling parameters change signs during the evolution and emphasize the role these systems playing in the second order phase transition. In the case of a time-dependent harmonic oscillator or an ensemble of such oscillators with a positive time-dependent frequency squared, the Fock space consists of the number states which are the exact quantum states of the Schrödinger equation [26,31]. The density operator is constructed in terms of the annihilation and creation operators that satisfy the LvN equation [28]. Hence the coherent, thermal and coherent-thermal states are found rather straightforwardly according to the standard technique of quantum mechanics. We further show that the same construction of the Fock space and density operator still holds for the time-dependent harmonic oscillator with a sign changing frequency squared. By studying some analytically solvable models we investigate how the instability grows. Another technical strong point of the LvN approach is that it can also be used for a time-dependent anharmonic oscillator. At the leading order the LvN approach is equipped with the time-dependent annihilation and creation operators, the vacuum state of which is already the time-dependent Gaussian state that minimizes the Dirac action [23,32]. The LvN approach thus provides one with a nonperturbative quantum description for the time-dependent anharmonic oscillator, too. We find the coherent, thermal and coherent-thermal states for the anharmonic oscillator with a polynomial potential and study the dynamics from their effective actions.

As field models for the second order phase transition, we consider first a free massive and then a self-interacting scalar field, the mass of which changes the sign in a finite time period through an external interaction. By studying analytically the exactly solvable model of the free scalar field, we show how the instability of long wavelength modes grows in time. The two-point thermal correlation function is expressed in terms of the classical solution for each mode that is already found in terms of a well-known function. The domain sizes are evaluated analytically by the steepest descent method and are shown to grow as $t^{1/4}$ during the quench and as the Cahn-Allen scaling relation $t^{1/2}$ after the completion of quench. Remarkably, the Cahn-Allen scaling relation shows a time-lag given by the cubic power of the quench period, which is absent in the instantaneous quench model [12,13]. However, both the instantaneous and the finite quench models have the same scaling relation given by the classical Cahn-Allen

equation [3], confirming the result of the instantaneous quench model [12,13]. The free scalar field model describes only the stage of spinodal instability from the unstable local maximum to the spinodal line. In the self-interacting field model, the back-reaction of an interacting term contributes positively to the frequencies of long wavelength modes and shuts off the exponential growth of instability after crossing the spinodal line at sufficiently later times.

The organization of this paper is as follows. It mainly consists of two parts: in the first part from Sec. II to Sec. V the LvN approach is elaborated to be applicable to phase transitions, and in the second part of Secs. VI and VII it is then applied to the second order phase transitions. In Sec. II we review the LvN approach to time-dependent quantum systems. In Sec. III we study the time-dependent harmonic oscillator and find the exact Fock space and the density operator. The density matrix is found and the nonequilibrium quantum dynamics is studied for the coherent, thermal and coherent-thermal states. In Sec. IV the LvN approach is applied to time-dependent inverted oscillators as a toy model for the second order phase transition. In Sec. V we extend the LvN approach to time-dependent anharmonic oscillators with polynomial potentials. The LvN approach leads to the nonperturbative Gaussian state at the leading order. We also show that the coherent state of the LvN approach recovers the result from the time-dependent mean-field or Hartree-Fock method. In Sec. VI we study an exactly solvable model of a free scalar field, which has a time-dependent mass coupling parameter and undergoes smoothly the second order phase transition for a finite quench period. The two-point thermal correlation function is analytically evaluated during the quench and after the completion of quench, and the scaling relations for the domain size are found. In Sec. VII we study a self-interacting scalar field that undergoes the time-dependent second order phase transition.

II. LIOUVILLE-VON NEUMANN (LVN) APPROACH

In this section we briefly review but emphasize the underlying assumptions of the LvN approach to time-dependent quantum systems introduced in Ref. [27]. A time-dependent system can not remain in the initial equilibrium or the instantaneous quasi-equilibrium because the density operator

$$\rho_{\mathrm{H}} = \frac{1}{Z_{\mathrm{H}}(t)} e^{-\beta \hat{H}(t)}, \tag{2.1}$$

where $\hat{H}(t)$ is a time-dependent Hamiltonian operator and Z_{H} is the partition function, does not satisfy the quantum LvN equation. This means that though the system starts in the initial thermal equilibrium, its final state can be far away from the initial one. Even an initial pure state evolves toward a final one which differs drastically from the initial one, and leads, for instance, to particle production [33–36]. The Fock or Hilbert space of such time-dependent system of a quantum field transforms unitarily inequivalently so that even the initial vacuum state evolves to a superposition of particle states at final times. Thus the nonequilibrium system described by time-dependent quantum Hamiltonian follows the evolution of a mixed state that is out of equilibrium and is characterized by time-dependent processes. To properly describe the nonequilibrium evolution we shall adopt two assumptions.

First, from a microscopic point of view we assume that the nonequilibrium system obeys the quantum law, *i.e.* the time-dependent Schrödinger or Tomonaga-Schwinger equation

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H}(t) |\Psi(t)\rangle. \tag{2.2}$$

Here $\hat{H}(t)$ is the time-dependent Hamiltonian of the system. This assumption is physically well-grounded since all the individual constituents of the system should obey the quantum law and the system as a whole should still obey the quantum law provided that all the interactions among the individuals are properly taken into account.

Second, from a statistical point of view it is assumed that even the nonequilibrium system obeys the time-dependent quantum Liouville-von Neumann (LvN) equation

$$i\hbar \frac{\partial}{\partial t} \hat{\rho}(t) + [\hat{\rho}(t), \hat{H}(t)] = 0. \tag{2.3}$$

The LvN equation has been used to find the density operator for equilibrium systems that are stationary. Now the nonequilibrium system with their time-dependent coupling parameters follows the same equation, so the density operator (2.1) that is directly defined in terms of the Hamiltonian itself does not necessarily satisfy the equation. In Sec. III we shall see how much the density operator satisfying Eq. (2.3) deviates from the instantaneous density operator (2.1) for time-dependent harmonic oscillators.

In the context of quantum mechanics a powerful canonical method was developed by Lewis and Riesenfeld [26]. They observed that any operator $\hat{\mathcal{O}}(t)$ satisfying the quantum LvN equation

$$i\hbar \frac{\partial}{\partial t} \hat{\mathcal{O}}(t) + [\hat{\mathcal{O}}(t), \hat{H}(t)] = 0, \tag{2.4}$$

can also be used to find the exact quantum states of Eq. (2.2). In fact, any exact quantum state is given by

$$|\Psi(t)\rangle = \sum_n c_n e^{\frac{i}{\hbar}\gamma_n(t)} |\lambda_n, t\rangle. \tag{2.5}$$

where

$$\begin{aligned} \hat{\mathcal{O}}(t)|\lambda_n, t\rangle &= \lambda_n |\lambda_n, t\rangle, \\ \gamma_n(t) &= \int dt \langle \lambda_n, t| \Big(i\hbar \frac{\partial}{\partial t} - \hat{H}(t) \Big) |\lambda_n, t\rangle. \end{aligned} \tag{2.6}$$

Another useful property of LvN approach is the linearity of the LvN equation. In fact, a product $\hat{\mathcal{O}}_1(t)\hat{\mathcal{O}}_2(t)$ satisfies Eq. (2.4) whenever $\hat{\mathcal{O}}_1(t)$ and $\hat{\mathcal{O}}_2(t)$ satisfy Eq. (2.4). Therefore it holds that any analytic functional $F[\mathcal{O}(t)]$ satisfies Eq. (2.4) provided that $\mathcal{O}(t)$ satisfies the same equation. In particular, we can still use $\hat{\mathcal{O}}(t)$ to define the density operator for the time-dependent system

$$\hat{\rho}_{\mathcal{O}}(t) = \frac{e^{-\beta \hat{\mathcal{O}}(t)}}{Z_{\mathcal{O}}(t)}, \quad Z_{\mathcal{O}}(t) = \mathrm{Tr}[e^{-\beta \hat{\mathcal{O}}(t)}]. \tag{2.7}$$

Here β is a free parameter and will be identified with the inverse temperature only for the equilibrium system, in which the Hamiltonian itself satisfies Eq. (2.4) and is used to define the standard density operator (2.1).

III. TIME-DEPENDENT OSCILLATOR

As a simple nonequilibrium system we begin with an ensemble of time-dependent harmonic oscillators. This system carries more meaning than being merely a toy model because most of systems with some exception such as the massless Φ^4-theory, either in equilibrium or in nonequilibrium, can be approximated by a quadratic Hamiltonian around zero-force points, stable or unstable. In the technical aspect the time-dependent oscillator can be exactly solved in terms of its classical solution. We now focus on a general oscillator with a time-dependent mass and frequency squared

$$\hat{H}(t) = \frac{1}{2m(t)}\hat{p}^2 + \frac{m(t)}{2}\omega^2(t)\hat{q}^2, \tag{3.1}$$

where $\omega^2(t)$ is allowed to change the sign during a phase transition. The LvN approach will be employed below to find the exact Fock space and to construct the coherent, thermal and coherent-thermal states.

A. Fock Space

The key idea of the LvN approach to the Hamiltonian (3.1) is to require a pair of operators [27,31]

$$\begin{aligned} \hat{a}(t) &= i\Big(u^*(t)\hat{p} - m(t)\dot{u}^*(t)\hat{q}\Big), \\ \hat{a}^\dagger(t) &= -i\Big(u(t)\hat{p} - m(t)\dot{u}(t)\hat{q}\Big), \end{aligned} \tag{3.2}$$

to satisfy the LvN equation (2.4). This results in the classical equation of motion for the complex auxiliary variable u

$$\ddot{u}(t) + \frac{\dot{m}(t)}{m(t)}\dot{u}(t) + \omega^2(t)u(t) = 0. \tag{3.3}$$

Note that these operators depend explicitly on time through $u(t)$ and are hermitian conjugate to each other. Further, these operators can be made the annihilation and creation operators with the standard commutation relation for all the times

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1. \tag{3.4}$$

The above commutation relation is guaranteed by the wronskian condition

$$\hbar m(t)\Big(\dot{u}^*(t)u(t) - \dot{u}(t)u^*(t)\Big) = i. \tag{3.5}$$

A comment is in order. Nothing prevents one from using these operators for an inverted time-dependent oscillator as far as Eq. (3.5) is satisfied. The inverted oscillator will be treated in detail in Sec. IV in the context of second order phase transitions.

¿From the argument in Sec. II, using $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ one may construct two particular operators that also satisfy Eq. (2.4): the number and the density operator. By defining the number operator in the usual way

$$\hat{N}(t) = \hat{a}^\dagger(t)\hat{a}(t), \tag{3.6}$$

one finds the Fock space consisting of the time-dependent number states

$$\hat{N}(t)|n,t\rangle = n|n,t\rangle. \tag{3.7}$$

The vacuum state is the one that is annihilated by $\hat{a}(t)$ and the nth number state is obtained by applying $\hat{a}^\dagger(t)$ n-times to the vacuum state:

$$\begin{aligned} \hat{a}(t)|0,t\rangle &= 0, \\ |n,t\rangle &= \frac{(\hat{a}^\dagger(t))^n}{\sqrt{n!}}|0,t\rangle. \end{aligned} \tag{3.8}$$

In the coordinate representation the number state is given by (see Appendix A)

$$\Psi_n(q,t) = \left(\frac{1}{2\pi\hbar^2 u^*(t)u(t)}\right)^{1/4} \frac{1}{\sqrt{2^n n!}}\left(\frac{u(t)}{u^*(t)}\right)^n H_n\left(\frac{q}{\sqrt{2\hbar^2 u^*(t)u(t)}}\right)\exp\left[\frac{i}{2}\frac{m}{\hbar}\frac{\dot{u}^*(t)}{u^*(t)}q^2\right], \tag{3.9}$$

where H_n is the Hermite polynomial.

Equation (3.2) can be inverted to yield the position and momentum operators

$$\begin{aligned} \hat{q} &= \hbar\Big(u(t)\hat{a}(t) + u^*(t)\hat{a}^\dagger\Big), \\ \hat{p} &= \hbar m(t)\Big(\dot{u}(t)\hat{a}(t) + \dot{u}^*(t)\hat{a}^\dagger\Big). \end{aligned} \tag{3.10}$$

Hence the expectation value of the position and momentum with respect to each number state vanishes

$$\langle n,t|\hat{q}|n,t\rangle = \langle n,t|\hat{p}|n,t\rangle = 0. \tag{3.11}$$

The only nonvanishing expectation values come from even powers of the position or momentum. The quadratic power of the position and momentum has the expectation values

$$\begin{aligned} \langle n,t|\hat{q}^2|n,t\rangle &= \hbar^2 u^*(t)u(t)(2n+1), \\ \langle n,t|\hat{p}^2|n,t\rangle &= \hbar^2 m^2(t)\dot{u}^*(t)\dot{u}(t)(2n+1), \\ \langle n,t|(\hat{q}\hat{p}+\hat{p}\hat{q})|n,t\rangle &= \hbar^2 m(t)\Big(\dot{u}^*(t)u(t) + u^*(t)\dot{u}(t)\Big)(2n+1). \end{aligned} \tag{3.12}$$

The Hamiltonian thus has the expectation value

$$H_{\mathrm{n}}(t) = \langle n,t|\hat{H}(t)|n,t\rangle = \frac{\hbar^2}{2}m(t)\Big[\dot{u}^*(t)\dot{u}(t) + \omega^2(t)u^*(t)u(t)\Big](2n+1). \tag{3.13}$$

The prominent advantages of using $\hat{a}^\dagger(t)$ and $\hat{a}(t)$ appear when one tries, by using the standard technique of quantum mechanics, to construct other quantum states such as the coherent, thermal and coherent-thermal states.

B. Coherent State

The coherent state is a particularly useful quantum state in quantum field theory for phase transitions or nonequilibrium dynamics. It may be used to obtain the effective potential for the classical background as on order parameter with contributions from quantum fluctuations. By treating $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ as the annihilation and creation operators, we follow the definition of the coherent state for the time-independent oscillator [37]. The coherent state is defined as an eigenstate of $\hat{a}(t)$:

$$\hat{a}(t)|\alpha, t\rangle = \alpha|\alpha, t\rangle, \tag{3.14}$$

where α is a complex constant. The coherent state can be treated algebraically by introducing a displacement operator

$$\hat{D}(\alpha) = e^{-\alpha\hat{a}^\dagger(t)+\alpha^*\hat{a}(t)}, \tag{3.15}$$

which is unitary

$$\hat{D}(\alpha)\hat{D}^\dagger(\alpha) = \hat{D}^\dagger(\alpha)\hat{D}(\alpha) = \hat{I}. \tag{3.16}$$

Moreover, the displacement operator translates by constants the annihilation and creation operators through the unitary transformation

$$\hat{D}(\alpha)\hat{a}(t)\hat{D}^\dagger(\alpha) = \hat{a}(t) + \alpha, \quad \hat{D}(\alpha)\hat{a}^\dagger(t)\hat{D}^\dagger(\alpha) = \hat{a}(t) + \alpha^*, \tag{3.17}$$

and the inverse unitary transformation

$$\hat{D}^\dagger(\alpha)\hat{a}(t)\hat{D}(\alpha) = \hat{a}(t) - \alpha, \quad \hat{D}^\dagger(\alpha)\hat{a}^\dagger(t)\hat{D}(\alpha) = \hat{a}(t) - \alpha^*. \tag{3.18}$$

Hence one sees from Eq. (3.17) that the coherent state results from the unitary transformation of the vacuum state

$$\begin{aligned} |\alpha, t\rangle &= \hat{D}^\dagger(\alpha)|0, t\rangle \\ &= e^{-\frac{\alpha^*\alpha}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n, t\rangle. \end{aligned} \tag{3.19}$$

It has been known since Schrödinger that the coherent state gives rise to a classical field for the time-independent oscillator [38]. In our time-dependent case, from the expectation values of the position and momentum with respect to the coherent state

$$\begin{aligned} q_c(t) &= \langle\alpha, t|\hat{q}|\alpha, t\rangle = \hbar\Big(\alpha u(t) + \alpha^* u^*(t)\Big), \\ p_c(t) &= \langle\alpha, t|\hat{p}|\alpha, t\rangle = \hbar m(t)\Big(\alpha\dot{u}(t) + \alpha^*\dot{u}^*(t)\Big), \end{aligned} \tag{3.20}$$

one sees that q_c indeed satisfies the classical equation of motion (3.3), because $u(t)$ has already satisfied Eq. (3.3) and α is a constant. Besides, q_c is real and $p_c = m(t)\dot{q}_c$, so one may identify q_c and p_c with the classical position and momentum.

In the above the coherent state has been constructed from the exact Fock space. There is another method to find the coherent state based on the minimization of action [18]. In

contrast with the LvN approach in which the operators (3.2) are required to satisfy the LvN equation, one regards u as a free parameter, works on the u-parameter Fock space and minimizes the Hamiltonian expectation value with respect to the coherent state. To show the method in detail, take the Hamiltonian expectation value with respect to the coherent state

$$H_{\rm C}(t) = \langle \alpha, t | \hat{H}(t) | \alpha, t \rangle = H_c(t) + H_{\rm V}(t), \tag{3.21}$$

which consists of the classical part

$$H_c(t) = \frac{1}{2m(t)} p_c^2 + \frac{m(t)}{2} \omega^2(t) q_c^2, \tag{3.22}$$

and the vacuum fluctuation part given in Eq. (3.13) with $n = 0$

$$H_{\rm V}(t) = \frac{\hbar^2}{2} m(t) \Big[\dot{u}^*(t) \dot{u}(t) + \omega^2(t) u^*(t) u(t) \Big]. \tag{3.23}$$

By writing the complex u in a polar form

$$u(t) = \frac{\xi(t)}{\sqrt{\hbar}} e^{-i\theta(t)}, \tag{3.24}$$

in terms of which Eq. (3.5) becomes $\dot{\theta} = 1/(2m\xi^2)$, and by introducing $p_\xi = m(t)\dot{\xi}$, one obtains the effective Hamiltonian

$$H_{\rm C}(t) = \frac{1}{2m(t)} p_c^2 + \frac{m(t)}{2} \omega^2(t) q_c^2 + \hbar \left[\frac{1}{2m(t)} p_\xi^2 + \frac{m(t)}{2} \omega^2(t) \xi^2 + \frac{1}{8m(t)\xi^2} \right]. \tag{3.25}$$

The last term in the square bracket of Eq. (3.25) has the same form as the angular momentum for a particle having rotational symmetry in two dimensions, but its origin is rooted on the condition (3.5) from quantization. Thus the effective Hamiltonian from the coherent state is equivalent to a two-dimensional Hamiltonian, which consists of the classical and quantum fluctuation parts. The variables q_c and ξ are independent and the Hamilton equations are for q_c

$$\begin{aligned} \frac{dq_c}{dt} &= \frac{\partial H_{\rm C}}{\partial p_c} = \frac{1}{m(t)} p_c, \\ \frac{dp_c}{dt} &= -\frac{\partial H_{\rm C}}{\partial q_c} = -m(t)\omega^2(t) q_c, \end{aligned} \tag{3.26}$$

and for ξ

$$\begin{aligned} \frac{d\xi}{dt} &= \frac{\partial (H_{\rm C}/\hbar)}{\partial p_\xi} = \frac{1}{m(t)} p_\xi, \\ \frac{dp_\xi}{dt} &= -\frac{\partial (H_{\rm C}/\hbar)}{\partial \xi} = -m(t)\omega^2(t)\xi + \frac{1}{4m(t)\xi^3}. \end{aligned} \tag{3.27}$$

It is then easy to show that the Hamilton equations (3.27) equal to the second order equation

$$\ddot{\xi}(t) + \frac{\dot{m}(t)}{m(t)}\dot{\xi}(t) + \omega^2(t)\xi - \frac{1}{4m^2(t)\xi^3} = 0, \qquad (3.28)$$

and that Eq. (3.28) is nothing but the equation (3.3) when u has the form (3.24) and satisfies the condition (3.5). Hence the minimization of the effective action gives the identical result as the LvN approach. Still another method is the mean field approach, in which the position and momentum are divided into a classical background and a fluctuation part

$$q = q_c + q_f, \quad p = p_c + p_f. \qquad (3.29)$$

Then the total Hamiltonian is composed of three parts: the classical background and fluctuation parts

$$H(t) = H_c(t) + H_f(t) + H_{int}(t), \qquad (3.30)$$

where

$$H_f(t) = \frac{1}{2m(t)}p_f^2 + \frac{m(t)}{2}\omega^2(t)q_f^2 \qquad (3.31)$$

is the fluctuation Hamiltonian, and

$$H_{int}(t) = \frac{1}{m(t)}p_c p_f + m(t)\omega^2(t)q_c q_f \qquad (3.32)$$

is the interaction Hamiltonian between the classical background and the fluctuation. And then quantize the fluctuation Hamiltonian (3.31) according to the method in the previous subsection but keep the classical one unquantized. Since the last two terms proportional to q_f and p_f have the zero expectation value, the expectation value of the total Hamiltonian (3.30) with respect to the vacuum state of the fluctuation Hamiltonian (3.31) yields exactly the effective Hamiltonian (3.25). Therefore, it has been shown that the expectation value of the original Hamiltonian with respect to the coherent state is equivalent to the sum of the classical part (3.22) and the vacuum expectation value of the fluctuation part (3.31).

C. Thermal State and Density Matrix

The ensemble of time-dependent oscillators exhibits intrinsically nonequilibrium behaviors, so it does lose a rigorous physical meaning to attribute any thermal property to the density operator (2.1). However, the harmonic oscillator problem is exactly solvable, so even in the time-dependent oscillator case one may look for a density operator which is quadratic in the position and momentum, and then fix their variable coefficients to satisfy the LvN equation [39]. On the other hand, in the LvN approach we can still use the operators (3.2) that have already satisfied the LvN equation and define the density operator from them. But there still remains a free parameter to incorporate the initial thermal equilibrium. We study the physical meaning of the density operator and see how the initial thermal equilibrium evolves quantum mechanically .

By noting that $\hat{N}(t)$ satisfies Eq. (2.4), we define the density operator by

$$\hat{\rho}_{\rm T}(t) = \frac{1}{Z_N} e^{-\beta\hbar\omega_0(\hat{N}(t)+\frac{1}{2})}, \tag{3.33}$$

where β and ω_0 are free parameters and Z_N is the partition function given by

$$Z_N = \sum_{n=0}^{\infty} \langle n,t| e^{-\beta\hbar\omega_0(\hat{N}(t)+\frac{1}{2})} |n,t\rangle = \frac{1}{2\sinh(\frac{\beta\hbar\omega_0}{2})}. \tag{3.34}$$

It has the same form as the standard density operator, the time-independent annihilation and creation operators now being replaced by the time-dependent ones (3.2). So Eq. (3.33) includes the time-independent case as a special case by choosing $\beta = 1/(k_B T)$ and ω_0 the oscillator frequency. In the coordinate representation the density matrix is given by (see Appendix B)

$$\begin{aligned} \rho_{\rm T}(q', q, t) &= \frac{1}{Z_N} \sum_{n=0}^{\infty} \Psi_n(q', t)\Psi_n^*(q, t) e^{-\beta\hbar\omega_0(n+\frac{1}{2})} \\ &= \left[\frac{\tanh(\frac{\beta\hbar\omega_0}{2})}{2\pi\hbar^2 u^* u}\right]^{1/2} \exp\left[\frac{i}{4}\frac{m}{\hbar}\frac{d}{dt}\ln(u^* u)(q'^2 - q^2)\right] \\ &\quad \times \exp\left[-\frac{1}{8\hbar^2 u^* u}\left\{(q'+q)^2 \tanh(\frac{\beta\hbar\omega_0}{2}) + (q'-q)^2 \coth(\frac{\beta\hbar\omega_0}{2})\right\}\right]. \end{aligned} \tag{3.35}$$

Now the density matrix (3.35) can be compared with that for the time-independent oscillator and the density operator (2.1) for the instantaneous Hamiltonian. For that purpose we restrict our attention to the particular case, in which the mass is constant, $m(t) = m_0$, and $\omega^2(t)$ is positive (see Sec. IV for the sign changing case of $\omega^2(t)$) and slowly changing $|\dot{\omega}(t)/\omega(t)| \ll 1$. In that case we may look for the solution to Eq. (3.3) of the form

$$u(t) = \frac{1}{\sqrt{2\hbar m_0 \Omega(t)}} e^{-i\int \Omega(t)}, \tag{3.36}$$

where

$$\Omega^2(t) = \omega^2(t) + \frac{3}{4}\frac{\dot{\Omega}^2(t)}{\Omega^2(t)} - \frac{1}{2}\frac{\ddot{\Omega}(t)}{\Omega(t)}. \tag{3.37}$$

The adiabatic (WKB) solution is obtained by approximating $\Omega(t) \approx \omega(t)$. Then the density matrix (3.35) reduces to the adiabatic one

$$\begin{aligned} \rho_{\rm A}(q', q, t) &= \left[\frac{m_0\omega(t)\tanh(\frac{\beta\hbar\omega_0}{2})}{\pi\hbar}\right]^{1/2} \exp\left[-\frac{i}{4}\frac{m_0}{\hbar}\frac{\dot{\omega}(t)}{\omega(t)}(q'^2 - q^2)\right] \\ &\quad \times \exp\left[-\frac{m_0\omega(t)}{4\hbar}\left\{(q'+q)^2 \tanh(\frac{\beta\hbar\omega_0}{2}) + (q'-q)^2 \coth(\frac{\beta\hbar\omega_0}{2})\right\}\right]. \end{aligned} \tag{3.38}$$

On the other hand, the density operator (2.1) for the instantaneous Hamiltonian has the matrix representation

$$\rho_{\mathrm{H}}(q', q, t) = \left[\frac{m_0\omega(t)\tanh(\frac{\beta\hbar\omega(t)}{2})}{\pi\hbar}\right]^{1/2}$$
$$\times \exp\left[-\frac{m_0\omega(t)}{4\hbar}\left\{(q'+q)^2\tanh(\frac{\beta\hbar\omega(t)}{2}) + (q'-q)^2\coth(\frac{\beta\hbar\omega(t)}{2})\right\}\right]. \quad (3.39)$$

In the case of the time-independent oscillator with $\omega(t) = \omega_0$, the density matrices (3.35) and (3.39) reduce further to the standard one [40]. The instantaneous density matrix is compared with the adiabatic one by taking the ratio

$$\frac{\rho_{\mathrm{H}}(q', q, t)}{\rho_{\mathrm{A}}(q', q, t)} = \left[\frac{\tanh(\frac{\beta\hbar\omega(t)}{2})}{\tanh(\frac{\beta\hbar\omega_0}{2})}\right]^{1/2} \exp\left[\frac{i}{4}\frac{m_0}{\hbar}\frac{\dot{\omega}(t)}{\omega(t)}(q'^2 - q^2)\right]$$
$$\times \exp\left[-\frac{m_0\omega(t)}{4\hbar}\left\{(q'+q)^2\frac{\sinh\left(\frac{\beta\hbar}{2}(\omega(t)-\omega_0)\right)}{\cosh(\frac{\beta\hbar\omega(t)}{2})\cosh(\frac{\beta\hbar\omega_0}{2})} - (q'-q)^2\frac{\sinh\left(\frac{\beta\hbar}{2}(\omega(t)-\omega_0)\right)}{\sinh(\frac{\beta\hbar\omega(t)}{2})\sinh(\frac{\beta\hbar\omega_0}{2})}\right\}\right]. \quad (3.40)$$

The second factor in Eq. (3.40) gives rise to a small phase factor because $\omega(t)$ is slowly varying. As far as $\omega(t)$ remains close to ω_0, the instantaneous density matrix (3.39) is close to the adiabatic one (3.38). Otherwise, the exact nonequilibrium evolution (3.38) is far away from the quasi-equilibrium one described by (3.39). This implies a significant deviation of the nonequilibrium evolution from the equilibrium one as $\omega(t)$ differs from ω_0 by a large amount.

Being mostly interested in phase transitions, we assume the system to start from an initial thermal equilibrium at early times. This requires that the oscillator have a constant frequency ω_0 at early times and β be fixed to the inverse temperature. As in the case of coherent state, the evolution of the initial thermal state can be found the effective Hamiltonian from this thermal state. From the expectation values (see the appendix of Ref. [28])

$$\langle \hat{q}^2\rangle_{\mathrm{T}} = \mathrm{Tr}\left[\hat{\rho}_{\mathrm{T}}(t)\hat{q}^2\right] = \hbar^2 u^*(t)u(t)\coth(\frac{\beta\hbar\omega_0}{2}),$$
$$\langle \hat{p}^2\rangle_{\mathrm{T}} = \mathrm{Tr}\left[\hat{\rho}_{\mathrm{T}}(t)\hat{p}^2\right] = \hbar^2 m^2(t)\dot{u}^*(t)\dot{u}(t)\coth(\frac{\beta\hbar\omega_0}{2}), \quad (3.41)$$

one obtains the effective Hamiltonian from the thermal state

$$H_{\mathrm{T}}(t) = \mathrm{Tr}\left[\hat{\rho}_{\mathrm{T}}(t)\hat{H}(t)\right] = \frac{\hbar^2}{2}m(t)\coth(\frac{\beta\hbar\omega_0}{2})\left[\dot{u}^*(t)\dot{u}(t) + \omega^2(t)u^*(t)u(t)\right]. \quad (3.42)$$

Once again by using the complex parameter u in the polar form (3.24) and by introducing $p_\xi = m(t)\dot{\zeta}$, one may rewrite the effective Hamiltonian as

$$H_{\mathrm{T}}(t) = \hbar\coth(\frac{\beta\hbar\omega_0}{2})\left[\frac{1}{2m(t)}p_\xi^2 + \frac{m(t)}{2}\omega^2(t)\zeta^2 + \frac{1}{8m(t)\xi^2}\right] = \coth(\frac{\beta\hbar\omega_0}{2})H_{\mathrm{V}}(t). \quad (3.43)$$

Since $\hbar\coth(\beta\hbar\omega_0/2)$ is constant, $H_{\mathrm{T}}(t)$ has the same Hamilton equations (3.28) as $H_{\mathrm{V}}(t)$. This is identical to the classical equation of motion (3.3) together with the boundary condition (3.5).

D. Coherent-Thermal State

A more general density operator that is at most quadratic in the position and momentum was introduced in Ref. [28]. It has the form

$$\hat{\rho}_{\rm C.T}(t) = \frac{1}{Z_{\rm C.T}(t)} \exp\Big[-\beta\Big\{\hbar\omega_0 \hat{a}^\dagger(t)\hat{a}(t) + \delta\hat{a}^\dagger(t) + \delta^*\hat{a}(t) + \epsilon_0\Big\}\Big], \tag{3.44}$$

where $Z_{\rm C.T}$ is a partition function. In fact, the density operator of Eq. (3.44) can be transformed into that of Eq. (3.33) by the unitary transformation

$$\hat{D}^\dagger(\alpha)\hat{\rho}_{\rm C}(t)\hat{D}(\alpha) = \hat{\rho}(t), \tag{3.45}$$

where $\hat{D}(\alpha)$ with $\alpha = \delta/(\hbar\omega_0)$ and $\epsilon_0 = (\hbar\omega_0/2) + (|\delta|^2/\hbar\omega_0)$. From the expectation values

$$\begin{aligned} \langle \hat{q}^2\rangle_{\rm C.T} &= {\rm Tr}\Big[\hat{\rho}_{\rm C.T}(t)\hat{q}^2\Big] = q_c^2 + \hbar^2 u^*(t)u(t)\coth(\frac{\beta\hbar\omega_0}{2}), \\ \langle \hat{p}^2\rangle_{\rm C.T} &= {\rm Tr}\Big[\hat{\rho}_{\rm C.T}(t)\hat{p}^2\Big] = p_c^2 + \hbar^2 m^2(t)\dot{u}^*(t)\dot{u}(t)\coth(\frac{\beta\hbar\omega_0}{2}), \end{aligned} \tag{3.46}$$

follows the effective Hamiltonian

$$H_{\rm C.T}(t) = H_c(t) + H_{\rm T}(t), \tag{3.47}$$

where H_c is the classical Hamiltonian. The effective Hamiltonian (3.47) has almost the same form as Eq. (3.25) except for the overall factor $\coth(\beta\hbar\omega_0/2)$, hence describes the same Hamilton equations (3.27).

In summary, what we have shown in this section is that the LvN approach provides us with the exact Fock space and various quantum states for time-dependent oscillators. The only auxiliary field necessary for that purpose is a complex solution u to the classical equation of motion (3.3) that satisfies the wronskian condition (3.5) from quantization. It has further been shown that the LvN approach is equivalent to the minimization principle for the effective action and to the mean field method. However, the advantage of the LvN approach lies in the manifest correspondence with quantum mechanics and quantum many-particle system and in the readiness to apply their standard techniques. For instance, the density matrix (3.35) has been found, which has many features similar with the standard one.

IV. INVERTED HARMONIC OSCILLATOR

As a toy model for the second order phase transition, let us consider the time-dependent harmonic oscillator

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2(t)\hat{q}^2, \tag{4.1}$$

where $\omega^2(t)$ has the asymptotic value $\omega_i^2(>0)$ far before and $-\omega_f^2(<0)$ far after the quench. The conspicuous point of the model is the sign change of $\omega^2(t)$. At earlier times before the

quench the oscillator executes a stable motion about $q = 0$, the global minimum, but after the quench the potential is inverted and $q = 0$ becomes an unstable configuration.

At earlier times far before the phase transition, the complex solution to Eq. (3.3) satisfying Eq. (3.5) is given by

$$u_i(t) = \frac{e^{-i\omega_i t}}{\sqrt{2\hbar\omega_i}}. \tag{4.2}$$

According to Eq. (3.2), the Fock space is now constructed from the annihilation and creation operators

$$\begin{aligned} \hat{a}(t) &= \frac{e^{i\omega_i t}}{\sqrt{2\hbar\omega_i}}\Bigl(i\hat{p} + \omega_i\hat{q}\Bigr) = e^{i\omega_i t}\hat{a}_0, \\ \hat{a}^\dagger(t) &= \frac{e^{-i\omega_i t}}{\sqrt{2\hbar\omega_i}}\Bigl(-i\hat{p} + \omega_i\hat{q}\Bigr) = e^{-i\omega_i t}\hat{a}_0^\dagger. \end{aligned} \tag{4.3}$$

Note that $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ differ from the standard ones $\hat{a}_0$ and $\hat{a}_0^\dagger$ only by phase factors. Though the Hamiltonian has the standard representation

$$\hat{H}_i = \hbar\omega_i\Bigl(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\Bigr) = \hbar\omega_i\Bigl(\hat{a}_0^\dagger\hat{a}_0 + \frac{1}{2}\Bigr), \tag{4.4}$$

the phase factors are necessary for $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ to satisfy the LvN equation. Hence the vacuum expectation value is given by the well-known result

$$H_{\rm V} = \frac{1}{2}\hbar\omega_i, \tag{4.5}$$

and the coherent state (3.19) yields

$$H_{\rm C} = \frac{1}{2}p_c^2 + \frac{1}{2}\omega_i^2 q_c^2 + \frac{1}{2}\hbar\omega_i. \tag{4.6}$$

Now the density operator (3.33) reduces to the standard one

$$\hat{\rho}_{i,\rm T} = \frac{1}{Z_N} e^{-\beta\hbar\omega_i(\hat{a}_0^\dagger\hat{a}_0 + \frac{1}{2})}, \tag{4.7}$$

after identifying $\omega_0 = \omega_i$ and $\beta = 1/(k_B T)$, and leads to the Hamiltonian expectation value

$$H_{i,\rm T} = \frac{1}{2}\hbar\omega_i \coth(\frac{\beta\hbar\omega_i}{2}). \tag{4.8}$$

On the other hand, at later times far after the quench, the solution to Eq. (3.3) is given by

$$\begin{aligned} u_f(t) &= \frac{1}{\sqrt{2\hbar}}\Bigl[C_1(\omega_i,\omega_f)\cosh(\omega_f t) - iC_2(\omega_i,\omega_f)\sinh(\omega_f t)\Bigr] \\ &= \frac{1}{2\sqrt{2\hbar}}\Bigl[\Bigl(C_1(\omega_i,\omega_f) - iC_2(\omega_i,\omega_f)\Bigr)e^{\omega_f t} + \Bigl(C_1(\omega_i,\omega_f) + iC_2(\omega_i,\omega_f)\Bigr)e^{-\omega_f t}\Bigr], \end{aligned} \tag{4.9}$$

where $C_j, j = 1, 2$ depend on the intermediate process toward the final state. Remarkably, the vacuum and thermal expectation values vanish:

$$H_{f,\mathrm{V}} = H_{f,\mathrm{T}} = 0, \tag{4.10}$$

since the kinetic and potential energies contribute equally. Whereas the expectation value with respect to the coherent state (3.19) and the general density operator (3.44) takes the form

$$H_{f,\mathrm{C}} = H_{f,\mathrm{C.T}} = \frac{1}{2}p_c^2 - \frac{1}{2}\omega_f^2 q_c^2. \tag{4.11}$$

This implies physically that as the system undergoes the phase transition out of equilibrium, quantum effects vanish and it becomes classical. Large uncertainty has been suggested as one of the criteria on the classicality [41]

$$(\Delta q)(\Delta p) = \frac{\hbar}{2}\Big[\omega_f C_1^2(\omega_i, \omega_f)\cosh^2(\omega_f t) + \omega_f C_1^2(\omega_i, \omega_f)\sinh^2(\omega_f t)\Big]. \tag{4.12}$$

Though the oscillator starts with the minimum uncertainty given by Eq. (4.2), its uncertainty increases exponentially and it becomes eventually classical after the completion of quench. When Eq. (4.9) is substituted into Eq. (3.35), the density matrix depends on the intermediate process and spreads as $\sqrt{u_f^*(t)u_f(t)}$. Therefore the quintessence of second order phase transitions lies in the whole process how systems evolve out of equilibrium from their initial equilibrium.

To show explicitly how the nonequilibrium dynamics depends on the intermediate processes, we consider two exactly solvable models. The first model, which is the zero mode of the free scalar model in Sec. VI, is an oscillator describing a finite smooth quench with the mass

$$m^2(t) = m_1^2 - m_0^2 \tanh\Big(\frac{t}{\tau}\Big). \tag{4.13}$$

The mass has $m_i^2 = m_0^2 + m_1^2$ at earlier times ($t = -\infty$) and has $-m_f^2 = -m_0^2 + m_1^2 < 0$ at later times ($t = \infty$). τ measures the quench rate, *i.e.* the rate of change of mass. The instantaneous quench corresponds to the ($\tau = 0$)-limit. The solution to Eq. (3.3) is found

$$u(t) = \frac{e^{-m_i t}}{\sqrt{2\hbar m_i}} {}_2F_1\Big(-\frac{\tau}{2}(im_i - m_f), -\frac{\tau}{2}(im_i + m_f); 1 - i\tau m_i; -e^{2t/\tau}\Big). \tag{4.14}$$

At earlier times the solution (4.14) has the correct asymptotic form

$$u_i(t) = \frac{e^{-im_i t}}{\sqrt{2\hbar m_i}}. \tag{4.15}$$

On the other hand, at later times the asymptotic form of Eq. (4.14) becomes [42]

$$\begin{aligned} u_f(t) = {} & \left[\frac{1}{\sqrt{2\hbar m_i}} \frac{(-1)\Gamma(1 - im_i\tau)\Gamma(m_f\tau)}{\frac{\tau}{2}(im_i - m_f)\Gamma^2(-\frac{\tau}{2}(im_i - m_f))}\right] e^{m_f t} \\ & + \left[\frac{1}{\sqrt{2\hbar m_i}} \frac{(-1)\Gamma(1 - im_i\tau)\Gamma(m_f\tau)}{\frac{\tau}{2}(im_i + m_f)\Gamma^2(-\frac{\tau}{2}(im_i + m_f))}\right] e^{-m_f t}. \end{aligned} \tag{4.16}$$

Two points are observed: the initial solution branches into an unstable growing and a decaying mode as expected and the coefficients C_1, C_2 of Eq. (4.9) depend on the the mass parameters m_i, m_f and the quench rate τ. In other words, the final asymptotic state of nonequilibrium evolution depends on the intermediate process.

The next model describes various quench processes and exhibits how the stable mode far before the quench branches into the unstable growing and decaying modes during the quench. Without loss of generality the nonequilibrium phase transition is assumed to take place through the time-dependent frequency (mass) squared

$$\omega^2(t) = \begin{cases} \omega_i^2, & t_i > t, \\ \omega_i^2 \left(\frac{t_0 - t}{t_0 - t_i}\right)^{(2l_1+1)/(2l_2+1)}, & t_0 > t > t_i, \\ -\omega_i^2 \left(\frac{t - t_0}{t_0 - t_i}\right)^{(2l_1+1)/(2l_2+1)}, & t_f > t > t_0, \\ -\omega_f^2 \equiv -\omega_i^2 \left(\frac{t_f - t_0}{t_0 - t_i}\right)^{(2l_1+1)/(2l_2+1)}, & t > t_f, \end{cases} \tag{4.17}$$

where l_1 and l_2 are non-negative integers. Here $t_0 - t_i$ adjusts the rate of and $t_f - t_0$ the duration of the quench. The particular case of $l_1 = l_2 = 0$ is used as the finite linear quench model [13,16]. Before the time t_0, the system maintains the symmetry about $q = 0$, the minimum of the potential. But as time goes on after t_0, $q = 0$ remains no longer the true minimum of the system and the symmetry is broken. The particular form of the power-law in Eq. (4.17) is chosen to allow an analytical continuation of $\omega^2(t)$ for changing the sign and to make its derivatives also continuous. Before $t_i > t$, the solution is given by Eq. (4.2). During $t_0 > t > t_i$, the solution is given by a linear superposition of Hankel functions

$$u(t) = D_1 z^\nu H_\nu^{(2)}(z) + D_2 z^\nu H_\nu^{(1)}(z), \tag{4.18}$$

where

$$z = 2\nu\omega_i(t_0 - t_i)\left(\frac{t_0 - t}{t_0 - t_i}\right)^{1/2\nu}, \quad \nu = \frac{2l_2 + 1}{2l_1 + 4l_2 + 3}. \tag{4.19}$$

Here $H_\nu^{(2)}$ and $H_\nu^{(1)}$ are positive and negative frequency solutions, respectively. The constants D_1 and D_2 are determined by continuity of $u(t)$ and $\dot{u}(t)$ across t_i:

$$\begin{aligned} D_1 &= \frac{e^{-i\omega_i t_i}}{\sqrt{2\hbar\omega_i}} \frac{\pi}{4z_i^\nu}\left[-iz_i \frac{d}{dz_i} H_\nu^{(1)}(z_i) - \Big(\omega_i - i\frac{1}{2(t_0 - t_i)}\Big) H_\nu^{(2)}(z_i)\right], \\ D_2 &= \frac{e^{-i\omega_i t_i}}{\sqrt{2\hbar\omega_i}} \frac{\pi}{4z_i^\nu}\left[iz_i \frac{d}{dz_i} H_\nu^{(2)}(z_i) + \Big(\omega_i - i\frac{1}{2(t_0 - t_i)}\Big) H_\nu^{(1)}(z_i)\right], \end{aligned} \tag{4.20}$$

where

$$z_i = 2\nu\omega_i(t_0 - t_i). \tag{4.21}$$

Beyond the quench time t_0, it is necessary to do carefully the analytic continuation and to take a suitable Riemann sheet so that $\omega^2(t)$ and its derivatives are to be continuous from $\omega^2 > 0$ to $\omega^2 < 0$ across t_0. The analytic continuation of the solution (4.18) yields

$$u(t) = \frac{D_1}{2}\tilde{z}^{\nu}\left[e^{i3\pi(l_2+\frac{1}{2})}H_{\nu}^{(2)}(i\tilde{z}) + e^{i\pi(l_2+\frac{1}{2})}H_{\nu}^{(2)}(-i\tilde{z})\right]$$
$$+\frac{D_2}{2}\tilde{z}^{\nu}\left[e^{i3\pi(l_2+\frac{1}{2})}H_{\nu}^{(1)}(i\tilde{z}) + e^{i\pi(l_2+\frac{1}{2})}H_{\nu}^{(1)}(-i\tilde{z})\right], \tag{4.22}$$

where

$$\tilde{z} = 2\nu\omega_i(t_0 - t_i)\left(\frac{t - t_0}{t_0 - t_i}\right)^{1/2\nu}. \tag{4.23}$$

At later times ($\tilde{z} \gg 1$) during the quench, the solution (4.22) has the asymptotic form [43]

$$u_f(t) = \sqrt{\frac{1}{2\pi}}\tilde{z}^{\nu-\frac{1}{2}}e^{\tilde{z}}\left[D_1 e^{i\pi\left(3l_2+\frac{\nu}{2}+\frac{3}{2}\right)} + D_2 e^{i\pi\left(l_2-\frac{\nu}{2}-\frac{1}{2}\right)}\right]. \tag{4.24}$$

Thus the stable mode of the oscillating solution (4.2) branches into the growing mode (4.24) which dominates during various quench processes and into the decaying mode which contributes negligibly to the correlation functions. The asymptotic solution (4.24) also depends on the intermediate processes through t_0, t_i, t_f and l_1, l_2.

V. TIME-DEPENDENT ANHARMONIC OSCILLATOR

In this section we extend the formalism developed in Sec. III to the time-dependent anharmonic oscillator with the Hamiltonian

$$H(t) = \frac{p^2}{2m(t)} + m(t)V(q), \tag{5.1}$$

where

$$V(q) = \frac{\lambda_{2n}(t)}{(2n)!}q^{2n}. \tag{5.2}$$

Though the potential of a power law is assumed, the formalism can readily be generalized to any polynomial and analytic potential. In the time-independent case (λ_{2n} = constant), the variational perturbation method has been introduced as one of the powerful methods to find the Hilbert space [44]. The vacuum state in this approach is the Gaussian wave functional that minimizes the effective action. The excited states are then obtained from the vacuum state just as number states of a harmonic oscillator are obtained from the ground state. Though these states can be calculated explicitly in terms a complex solution to the classical equation for motion, they are in fact equivalent to those from the mean-field method.

However, there have been some attempts to go beyond the Gaussian state. In Ref. [45] a scheme was proposed to find the operators, which generalize the annihilation and creation operators and satisfy the LvN equation (2.4), to all the orders of coupling constant in the time-independent case and to the first order in the time-dependent case. In particular, the generalized annihilation and creation operators for the time-independent oscillator with a quartic potential satisfy a q-deformed algebra rather than the standard commutation

relation [46], from which follows an algebraic construction of excited states and energy spectra beyond the variational Gaussian approximation. It would be interesting to find such an algebraic structure for interacting quantum fields, which may shed some light on the nonperturbative method beyond the mean-field method. Also it would be interesting to compare this scheme with other nonperturbative methods in the time-independent case such as the perturbative expansion method around the Gaussian effective action [47] and the time-dependent variational method [48]. But we shall not pursue further this issue in this paper.

A. Fock Space

In the case of time-dependent anharmonic oscillators, the LvN approach searches for the annihilation and creation operators that are still linear in the position and momentum and satisfy the LvN equation (2.4):

$$\hat{A}(t) = i\Big(v^*(t)\hat{p} - m(t)\dot{v}^*(t)\hat{q}\Big),$$
$$\hat{A}^\dagger(t) = -i\Big(v(t)\hat{p} - m(t)\dot{v}(t)\hat{q}\Big). \tag{5.3}$$

One then requires them to satisfy the LvN equation (2.4), leading to the equation

$$\ddot{v}(t)\hat{q} + \frac{\dot{m}(t)}{m(t)}\dot{v}(t)\hat{q} + v(t)\frac{\delta V(\hat{q})}{\delta \hat{q}} = 0, \tag{5.4}$$

and further differentiates functionally with respect to $\hat{q}$ and takes the vacuum expectation value of the resultant equation

$$\ddot{v}(t) + \frac{\dot{m}(t)}{m(t)}\dot{v}(t) + \langle 0,t|\frac{\delta^2 V(\hat{q})}{\delta \hat{q}^2}|0,t\rangle v(t) = 0. \tag{5.5}$$

Here the vacuum state is annihilated by $\hat{A}(t)$

$$\hat{A}(t)|0,t\rangle = 0. \tag{5.6}$$

One makes $\hat{A}(t)$ and $\hat{A}^\dagger(t)$ the annihilation and creation operators, respectively, by imposing the standard commutation relation for all times

$$[\hat{A}(t), \hat{A}^\dagger(t)] = 1. \tag{5.7}$$

Equation (5.7) is equivalent to the wronskian condition

$$\hbar m(t)\Big(\dot{v}^*(t)v(t) - \dot{v}(t)v^*(t)\Big) = i. \tag{5.8}$$

The Fock space consists of the number state obtained by applying $\hat{A}^\dagger(t)$ n-times to the vacuum state

$$|n,t\rangle = \frac{\Big(\hat{A}^\dagger(t)\Big)^n}{\sqrt{n!}}|0,t\rangle. \tag{5.9}$$

These number states are excited states and have the coordinate representation (3.9) now with $u(t)$ replaced by $v(t)$.

¿From the position and momentum operators expressed in terms of the annihilation and creation operators

$$\hat{q} = \hbar\Big(v(t)\hat{A}(t) + v^*(t)\hat{A}^\dagger(t)\Big),$$
$$\hat{p} = \hbar m(t)\Big(\dot{v}(t)\hat{A}(t) + \dot{v}^*(t)\hat{A}^\dagger(t)\Big), \tag{5.10}$$

follow the vacuum expectation values

$$\langle 0, t|\hat{q}^{2n}|0, t\rangle = \frac{(2n)!}{2^n n!}\Big[\hbar^2 v^*(t)v(t)\Big]^n,$$
$$\langle 0, t|\hat{p}^2|0, t\rangle = \hbar^2 m^2(t)\dot{v}^*(t)\dot{v}(t). \tag{5.11}$$

Then the classical equation of motion (5.5) becomes

$$\ddot{v}(t) + \frac{\dot{m}(t)}{m(t)}\dot{v}(t) + \frac{\lambda_{2n}(t)}{2^{n-1}(n-1)!}\Big[\hbar^2 v^*(t)v(t)\Big]^{n-1} v(t) = 0. \tag{5.12}$$

There is another method to derive Eq. (5.12). By using the Wick-ordering

$$\hat{q}^{2n} = \sum_{k=0}^{n} \frac{(2n)!\hbar^{2n}}{2^k k!(2n-2k)!}\Big[v^*(t)v(t)\Big]^k : \Big[v(t)\hat{A}(t) + v^*(t)\hat{A}^\dagger(t)\Big]^{2(n-k)} :, \tag{5.13}$$

one obtains the Hamiltonian truncated at the quadratic order of $\hat{A}(t)$ and $\hat{A}(t)$

$$\hat{H}_{\rm G} = \frac{1}{2}\hbar^2 m(t)\Big[(\dot{v}(t)\hat{A}(t))^2 + 2\dot{v}^*(t)\dot{v}(t)\hat{A}^\dagger(t)\hat{A}(t) + (\dot{v}^*(t)\hat{A}^\dagger(t))^2\Big]$$
$$+ m(t)\frac{\hbar^{2n}\lambda_{2n}(t)}{2^n(n-1)!}[v^*(t)v(t)]^{n-1}\Big[(v(t)\hat{A}(t))^2 + 2v^*(t)v(t)\hat{A}^\dagger(t)\hat{A}(t) + (v^*(t)\hat{A}^\dagger(t))^2\Big], \tag{5.14}$$

where purely c-number terms are neglected. One then requires $\hat{A}^\dagger(t)$ and $\hat{A}(t)$ to satisfy the LvN equation (2.4) for the truncated Hamiltonian $\hat{H}_{\rm G}(t)$. Now the LvN equations for $\hat{A}^\dagger(t)$ and $\hat{A}(t)$ lead exactly to the equation of motion (5.12).

B. Effective Hamiltonians

Still another method to derive the equations of motion for v and q_c is the minimization principle for the effective action. For that purpose we consider a complex $v(t)$-parameter family of the Fock spaces constructed by the annihilation and creation operators (5.3) and take the Hamiltonian expectation value with respect to various quantum states such as the vacuum, coherent, thermal and coherent-thermal states. We do not require $\hat{A}(t)$ and $\hat{A}^\dagger(t)$ to satisfy the LvN equation a priori, but minimize the action to determine the equation of motion for $v(t)$.

The first effective Hamiltonian is the vacuum expectation value

$$H_{\rm V}(t) = \langle 0, t|\hat{H}(t)|0, t\rangle = \frac{\hbar^2}{2}m(t)\dot{v}^*\dot{v} + m(t)\frac{\lambda_{2n}(t)}{2^n n!}\Big[\hbar^2 v^*(t)v(t)\Big]^n. \tag{5.15}$$

The next state under consideration is the coherent state, which is obtained by applying a displacement operator to the vacuum state,

$$|\alpha, t\rangle = \hat{D}^\dagger(\alpha)|0, t\rangle = e^{\alpha \hat{A}^\dagger(t) - \alpha^* \hat{A}(t)}|0, t\rangle. \tag{5.16}$$

Then the coherent state expectation value leads to the effective Hamiltonian

$$H_{\rm C}(t) = \langle \alpha, t|\hat{H}(t)|\alpha, t\rangle, \tag{5.17}$$

which, with the aid of Eq. (5.13), is decomposed into

$$H_{\rm C}(t) = H_c(t) + H_q(t). \tag{5.18}$$

Here $H_c(t)$ is the classical part

$$H_c(t) = \frac{p_c^2}{2m(t)} + m(t)\frac{\lambda_{2n}(t)}{(2n)!}q_c^{2n}, \tag{5.19}$$

and $H_q(t)$ denotes all the quantum contributions including the vacuum expectation value

$$H_q(t) = \frac{\hbar^2}{2}m(t)\dot{v}^*\dot{v} + m(t)\sum_{k=1}^{n}\frac{\lambda_{2n}(t)}{2^k k!(2n-2k)!}\Big[\hbar^2 v^*(t)v(t)\Big]^k q_c^{2(n-k)}, \tag{5.20}$$

where

$$q_c = \langle \alpha, t|\hat{q}|\alpha, t\rangle, \quad p_c = \langle \alpha, t|\hat{p}|\alpha, t\rangle. \tag{5.21}$$

The final state is the thermal state defined by the density operator

$$\hat{\rho}_{\rm T} = \frac{1}{Z_N}e^{-\beta\omega_0(\hat{A}^\dagger(t)\hat{A}(t) + \frac{1}{2})}, \tag{5.22}$$

with Z_N being the partition function. The density operator (5.22) leads to the effective Hamiltonian

$$H_{\rm T}(t) = {\rm Tr}\Big[\hat{\rho}_{\rm T}\hat{H}(t)\Big] = \frac{\hbar^2}{2}m(t)\dot{v}^*\dot{v} + m(t)\frac{\lambda_{2n}(t)}{2^n n!}\langle \hat{q}^2\rangle_{\rm T}^n, \tag{5.23}$$

where

$$\langle \hat{q}^2\rangle_{\rm T} = \hbar^2 v^* v \coth(\frac{\beta\omega_0}{2}). \tag{5.24}$$

Likewise, the density operator of the form (3.44) with $\hat{a}^\dagger(t)$ and $\hat{a}(t)$ replaced by $\hat{A}^\dagger(t)$ and $\hat{A}(t)$ leads to the effective Hamiltonian

$$H_{\rm C.T}(t) = H_c(t) + \frac{\hbar^2}{2}m(t)\coth(\frac{\beta\hbar\omega_0}{2})\dot{v}^*\dot{v} + m(t)\sum_{k=1}^{n}\frac{\lambda_{2n}(t)}{2^k k!(2n-2k)!}\langle \hat{q}^2\rangle_{\rm T}^k q_c^{2(n-k)}. \tag{5.25}$$

Note that Eqs. (5.15) and (5.18) are also obtained from Eqs. (5.23) and (5.25), respectively, by replacing $\hbar^2 v^* v \coth(\beta\omega_0/2)$ with $\hbar^2 v^* v$ or taking the zero-temperature limit ($\beta \to \infty$).

We now study the dynamics of the effective Hamiltonians. We mainly focus on the effective Hamiltonian (5.25) since Eq. (5.18) is the limiting case of Eq. (5.25) when $\beta \to \infty$, *i.e.*, $T \to 0$, and Eq. (5.15) is the limiting case of Eq. (5.18) when $q_c = p_c = 0$. The independent variables of the Hamiltonian (5.25) are (q_c, p_c), $(v, p_v = m\dot{v}^*)$ and $(v^*, p_v^* = m\dot{v})$. So we obtain the equation of motion for q_c

$$\ddot{q}_c + \frac{\dot{m}}{m}\dot{q}_c + \frac{\lambda_{2n}(t)}{(2n-1)!}q_c^{2n-1} + \sum_{k=1}^{n} \frac{\lambda_{2n}(t)}{2^k k!(2n-2k-1)!}\langle \hat{q}^2 \rangle_{\mathrm{T}}^k q_c^{2n-2k-1} = 0. \tag{5.26}$$

The equation of motion for v^* is given by

$$\ddot{v} + \frac{\dot{m}}{m}\dot{v} + \sum_{k=1}^{n} \frac{\lambda_{2n}(t)}{2^{k-1}(k-1)!(2n-2k)!}q_c^{2n-2k}\langle \hat{q}^2 \rangle_{\mathrm{T.}}^{k-1} v = 0, \tag{5.27}$$

and the complex conjugate of Eq. (5.27) is for v. The equations of motion from the effective Hamiltonian (5.18) is the limiting case of Eqs. (5.26) and (5.27) when $\langle \hat{q}^2 \rangle_{(\mathrm{T})} = \hbar^2 v^* v$:

$$\begin{aligned} \ddot{q}_c + \frac{\dot{m}}{m}\dot{q}_c + \frac{\lambda_{2n}(t)}{(2n-1)!}q_c^{2n-1} + \sum_{k=1}^{n} \frac{\lambda_{2n}(t)}{2^k k!(2n-2k-1)!}(\hbar^2 v^* v)^k q_c^{2n-2k-1} = 0, \\ \ddot{v} + \frac{\dot{m}}{m}\dot{v} + \sum_{k=1}^{n} \frac{\lambda_{2n}(t)}{2^{k-1}(k-1)!(2n-2k)!}q_c^{2n-2k}(\hbar^2 v^* v)^{k-1} v = 0. \end{aligned} \tag{5.28}$$

And the limiting case $q_c = 0$ of Eq. (5.28) is the equation for the effective Hamiltonian (5.15)

$$\ddot{v} + \frac{\dot{m}}{m}\dot{v} + \frac{\lambda_{2n}(t)}{2^{n-1}(n-1)!}(\hbar^2 v^* v)^{n-1} v = 0. \tag{5.29}$$

Note that Eq. (5.29) is identical to Eq. (5.12) from the LvN approach.

Or, by writing v in the polar form

$$v(t) = \frac{\zeta(t)}{\sqrt{\hbar}} e^{-i\theta(t)}, \tag{5.30}$$

and by introducing the momentum $p_\zeta = m(t)\dot{\zeta}$, the effective Hamiltonian (5.25) is rewritten as

$$\begin{aligned} H_{\mathrm{C.T}}(t) = \frac{p_c^2}{2m(t)} + m(t)\frac{\lambda_{2n}(t)}{(2n)!}q_c^{2n} + \hbar \coth(\frac{\beta\hbar\omega_0}{2})\left[\frac{p_\zeta^2}{2m(t)} + \frac{1}{8m(t)\zeta^2}\right] \\ + m(t)\sum_{k=1}^{n} \frac{\lambda_{2n}(t)}{2^k k!(2n-2k)!}\left[\hbar\zeta^2 \coth(\frac{\beta\hbar\omega_0}{2})\right]^k q_c^{2(n-k)}. \end{aligned} \tag{5.31}$$

Now the equation of motion for the classical field q_c is given by

$$\ddot{q}_c + \frac{\dot{m}}{m}\dot{q}_c + \frac{\lambda_{2n}(t)}{(2n-1)!}q_c^{2n-1} + \sum_{k=1}^{n} \frac{\lambda_{2n}(t)}{2^k k!(2n-2k-1)!}[\hbar\zeta^2 \coth(\frac{\beta\hbar\omega_0}{2})]^k q_c^{2n-2k-1} = 0, \tag{5.32}$$

and that for ζ by

$$\ddot{\zeta} + \frac{\dot{m}}{m}\dot{\zeta} - \frac{1}{4m^2\zeta^3} + \sum_{k=1}^{n} \frac{\lambda_{2n}(t)}{2^{k-1}(k-1)!(2n-2k)!} q_c^{2n-2k} [\hbar\zeta^2 \coth(\frac{\beta\omega_0}{2})]^{k-1}\zeta = 0. \tag{5.33}$$

The phase of $v(t)$ is obtained by the integration

$$\theta(t) = \int \frac{1}{2m(t)\zeta^2(t)}. \tag{5.34}$$

C. Coherent State vs. Hartree-Fock Method

In this subsection we show that the nonequilibrium dynamics obtained from the effective Hamiltonian (5.25) in the LvN approach recovers exactly the equations of motion from the mean field and Hartree-Fock methods.

First, the effective Hamiltonian from the coherent state can be obtained in another way. By dividing q and p into a classical background and a quantum fluctuation

$$q = q_c + q_f, \tag{5.35}$$

we obtain the expectation value for the Hamiltonian (5.1) with respect to the thermal state (5.22)

$$H_{\rm c.f.}(t) = \frac{p_c^2}{2m(t)} + \frac{\langle \hat{p}_f^2 \rangle_{\rm T}}{2m(t)} + m(t)\sum_{k=0}^{n} \frac{\lambda_{2n}(t)}{2^k k!(2n-2k)!} \langle \hat{q}_f^2 \rangle_{\rm T}^k q_c^{2(n-k)}. \tag{5.36}$$

where we have used

$$\langle \hat{q}^{2n} \rangle_{\rm T} = \sum_{k=0}^{n} \frac{(2n)!}{2^k k!(2n-2k)!} q_c^{2(n-k)} \left[\hbar^2 v^* v \coth(\frac{\beta\hbar\omega_0}{2})\right]^k, \tag{5.37}$$

and

$$\langle \hat{q}_f^{2n+1} \rangle_{\rm T} = 0 = \langle \hat{p}_f^{2n+1} \rangle_{\rm T}. \tag{5.38}$$

Noting that $k = 0$ term recovers the classical potential and

$$\langle \hat{p}_f^2 \rangle_{\rm T} = m^2\hbar^2 \dot{v}^* \dot{v} \coth(\frac{\beta\hbar\omega_0}{2}), \tag{5.39}$$

we can show that Eq. (5.36) coincides with Eq. (5.25). Therefore the coherent state leads exactly to the result from the mean-field method.

Second, the Hartree-Fock factorization theorem [21] leads to the effective Hamiltonian

$$H_{\rm H.F}(t) = \frac{\langle \hat{p}^2 \rangle_{\rm H.F}}{2m(t)} + m(t)\frac{\lambda_{2n}(t)}{(2n)!} \langle \hat{q}^{2n} \rangle_{\rm H.F}, \tag{5.40}$$

where

$$\begin{aligned} \langle \hat{p}^2 \rangle_{\rm H.F} &= p_c^2 + \hat{p}_f^2, \\ \langle \hat{q}^2 \rangle_{\rm H.F} &= q_c^2 + \hat{q}_f^2 \end{aligned} \tag{5.41}$$

and

$$\langle \hat{q}^{2n} \rangle_{\rm H.F} = \sum_{k=0}^{2n} \frac{(2n)!}{k!(2n-k)!} q_c^{2n-k} \hat{q}_f^k$$
$$= \sum_{k=0}^{n} \frac{(2n)!}{2^k k!(2n-2k)!} q_c^{2(n-k)} \Big[k \langle \hat{q}_f^2 \rangle_{\rm T}^{k-1} \hat{q}_f^2 - (k-1) \langle \hat{q}_f^2 \rangle_{\rm T}^{k} \Big]$$
$$+ \sum_{k=0}^{n-1} \frac{(2n)!}{2^k k!(2n-2k-1)!} q_c^{2n-2k-1} \langle \hat{q}_f^2 \rangle_{\rm T}^{k} \hat{q}_f. \tag{5.42}$$

for $n \geq 2$. The thermal expectation value of the equation of motion for q_c yields

$$\ddot{q}_c + \frac{\dot{m}}{m} \dot{q}_c + \sum_{k=0}^{n} \frac{\lambda_{2n}(t)}{2^k k!(2n-2k)!} \langle \hat{q}_f^2 \rangle_{\rm T}^{k} q_c^{2n-2k-1} = 0, \tag{5.43}$$

and for $\hat{q}_f$

$$\ddot{\hat{q}}_f + \frac{\dot{m}}{m} \dot{\hat{q}}_f + \sum_{k=0}^{n} \frac{\lambda_{2n}(t)}{2^{k-1}(k-1)!(2n-2k)!} \langle \hat{q}_f^2 \rangle_{\rm T}^{k-1} q_c^{2(n-k)} \hat{q}_f = 0. \tag{5.44}$$

Therefore it has been shown that Eqs. (5.43) and (5.44) are the same as Eqs. (5.26) and (5.27) from the coherent state representation.

VI. FREE SCALAR FIELD FOR PHASE TRANSITION

As a simple field model for the second order phase transition, we consider a free complex scalar field, the mass of which changes the sign during the quench.[1] The system is described by the Lagrangian density

$$\mathcal{L}(\mathbf{x}, t) = \dot{\Phi}^*(\mathbf{x}, t) \dot{\Phi}(\mathbf{x}, t) - \nabla \Phi^*(\mathbf{x}, t) \cdot \nabla \Phi(\mathbf{x}, t) - m^2(t) \Phi^*(\mathbf{x}, t) \Phi(\mathbf{x}, t). \tag{6.1}$$

Here the coupling parameter $m^2(t)$ is assumed to begin with an initial positive value before, to change the sign during, and to reach a final negative value after the quench. The Φ and Φ^* are treated as independent fields. The Hamiltonian is given by

$$H(t) = \int d^3\mathbf{x} \Big[\Pi^*(\mathbf{x}, t) \Pi(\mathbf{x}, t) + \nabla \Phi^*(\mathbf{x}, t) \cdot \nabla \Phi(\mathbf{x}, t) + m^2(t) \Phi^*(\mathbf{x}, t) \Phi(\mathbf{x}, t) \Big], \tag{6.2}$$

where

$$\Pi(\mathbf{x}, t) = \frac{\delta \mathcal{L}(\mathbf{x}, t)}{\delta \dot{\Phi}(\mathbf{x}, t)} = \dot{\Phi}^*(\mathbf{x}, t),$$
$$\Pi^*(\mathbf{x}, t) = \frac{\delta \mathcal{L}(\mathbf{x}, t)}{\delta \dot{\Phi}^*(\mathbf{x}, t)} = \dot{\Phi}(\mathbf{x}, t) \tag{6.3}$$

---

[1] The complex scalar field model may be related with the time-dependent Landau-Ginsburg theory provided that the free energy be interpreted as the Hamiltonian in this paper [16].

are conjugate momenta.

The field and momentum are Fourier-decomposed as

$$\Phi(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\phi_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}},$$
$$\Pi(\mathbf{x},t) = \dot{\Phi}^*(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\dot{\phi}^*_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}} \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3}\pi_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{6.4}$$

and their conjugates as

$$\Phi^*(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\phi^*_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}},$$
$$\Pi^*(\mathbf{x},t) = \dot{\Phi}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\dot{\phi}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}} \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3}\pi^*_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{6.5}$$

So space integrals of quadratic fields and momenta result in momentum integrals for the decoupled modes

$$\int d^3\mathbf{x}\Phi^*\Phi = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\phi^*_{\mathbf{k}}\phi_{\mathbf{k}},$$
$$\int d^3\mathbf{x}\Pi^*\Pi = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\pi^*_{\mathbf{k}}\pi_{\mathbf{k}},$$
$$\int d^3\mathbf{x}\nabla\Phi^*\cdot\nabla\Phi = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\mathbf{k}^2\phi^*_{\mathbf{k}}\phi_{\mathbf{k}}. \tag{6.6}$$

One then obtains the Hamiltonian as the sum of infinite number of time-dependent harmonic oscillators

$$H(t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\Big[\pi^*_{\mathbf{k}}\pi_{\mathbf{k}} + \Big(\mathbf{k}^2 + m^2(t)\Big)\phi^*_{\mathbf{k}}\phi_{\mathbf{k}}\Big]. \tag{6.7}$$

Canonical quantization is prescribed by imposing the commutation relations at equal times

$$[\hat{\phi}_{\mathbf{k}'}(t), \hat{\pi}_{\mathbf{k}}(t)] = i\hbar\delta_{\mathbf{k},\mathbf{k}'},$$
$$[\hat{\phi}^*_{\mathbf{k}'}(t), \hat{\pi}^*_{\mathbf{k}}(t)] = i\hbar\delta_{\mathbf{k},\mathbf{k}'}, \tag{6.8}$$

and all the other commutators vanish. Following Sec. III, we find the two pairs of the annihilation and creation operators (3.2) for each $\mathbf{k}$-mode,

$$\hat{a}_{\mathbf{k}}(t) = i\Big(\varphi^*_{\mathbf{k}}(t)\hat{\pi}^*_{\mathbf{k}} - \dot{\varphi}^*_{\mathbf{k}}(t)\hat{\phi}_{\mathbf{k}}\Big),$$
$$\hat{a}^\dagger_{\mathbf{k}}(t) = -i\Big(\varphi_{\mathbf{k}}(t)\hat{\pi}_{\mathbf{k}} - \dot{\varphi}_{\mathbf{k}}(t)\hat{\phi}^*_{\mathbf{k}}\Big), \tag{6.9}$$

and

$$\hat{a}^*_{\mathbf{k}}(t) = i\Big(\varphi^*_{\mathbf{k}}(t)\hat{\pi}_{\mathbf{k}} - \dot{\varphi}^*_{\mathbf{k}}(t)\hat{\phi}^*_{\mathbf{k}}\Big),$$
$$\hat{a}^{*\dagger}_{\mathbf{k}}(t) = -i\Big(\varphi_{\mathbf{k}}(t)\hat{\pi}^*_{\mathbf{k}} - \dot{\varphi}_{\mathbf{k}}(t)\hat{\phi}_{\mathbf{k}}\Big), \tag{6.10}$$

where $\varphi_{\mathbf{k}}$ and $\varphi^*_{\mathbf{k}}$ satisfy the same classical equation of motion

$$\ddot{\varphi}_{\mathbf{k}}(t) + \Big(\mathbf{k}^2 + m^2(t)\Big)\varphi_{\mathbf{k}}(t) = 0. \tag{6.11}$$

They further satisfy the standard commutation relations

$$[\hat{a}_{\mathbf{k}'}, \hat{a}^\dagger_{\mathbf{k}}] = \delta_{\mathbf{k},\mathbf{k}'}, \quad [\hat{a}^*_{\mathbf{k}'}, \hat{a}^{*\dagger}_{\mathbf{k}}] = \delta_{\mathbf{k},\mathbf{k}'}. \tag{6.12}$$

The Fock space for each mode can be constructed according to Sec. III. We consider two symmetric states: the vacuum and thermal states. The vacuum is the one annihilated by all the $\hat{a}_{\mathbf{k}}$ and $\hat{a}^*_{\mathbf{k}}$:

$$\hat{a}_{\mathbf{k}}(t)|0,t\rangle = 0, \quad \hat{a}^*_{\mathbf{k}}(t)|0,t\rangle = 0. \tag{6.13}$$

By inverting Eqs. (6.9) and (6.10) one expresses the fields as

$$\begin{aligned} \hat{\phi}_{\mathbf{k}} &= \hbar\Big(\varphi_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \dot{\varphi}_{\mathbf{k}}{}^*\hat{a}^{*\dagger}_{\mathbf{k}}\Big), \\ \hat{\phi}^*_{\mathbf{k}} &= \hbar\Big(\varphi_{\mathbf{k}}\hat{a}_{*\mathbf{k}} + \dot{\varphi}_{\mathbf{k}}{}^*\hat{a}^{\dagger}_{\mathbf{k}}\Big), \end{aligned} \tag{6.14}$$

from which follow the vacuum expectation values

$$\begin{aligned} \langle\hat{\Phi}^*\hat{\Phi}\rangle_{\mathrm{V}} &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\Big[\hbar^2\varphi^*_{\mathbf{k}}(t)\varphi_{\mathbf{k}}(t)\Big], \\ \langle\hat{\Pi}^*\hat{\Pi}\rangle_{\mathrm{V}} &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\Big[\hbar^2\dot{\varphi}^*_{\mathbf{k}}(t)\dot{\varphi}_{\mathbf{k}}(t)\Big]. \end{aligned} \tag{6.15}$$

The initial thermal state defined by the density operator for each mode

$$\begin{aligned} \hat{\rho}(t) = \prod_{\mathbf{k}}\hat{\rho}_{\mathbf{k}}(t) = \prod_{\mathbf{k}}\Bigg\{&\frac{1}{Z_{\mathbf{k}}}\exp\Big[-\beta\hbar\omega_{i,\mathbf{k}}\Big(\hat{a}^\dagger_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t) + \frac{1}{2}\Big)\Big] \\ &\times\frac{1}{Z^*_{\mathbf{k}}}\exp\Big[-\beta\hbar\omega_{i,\mathbf{k}}\Big(\hat{a}^{*\dagger}_{\mathbf{k}}(t)\hat{a}^*_{\mathbf{k}}(t) + \frac{1}{2}\Big)\Big]\Bigg\} \end{aligned} \tag{6.16}$$

leads to the thermal expectation values

$$\begin{aligned} \langle\hat{\Phi}^*\hat{\Phi}\rangle_{\mathrm{T}} = \mathrm{Tr}\Big[\hat{\rho}(t)\hat{\Phi}^*\hat{\Phi}\Big] &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\Big[\hbar^2\varphi^*_{\mathbf{k}}(t)\varphi_{\mathbf{k}}(t)\coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2})\Big], \\ \langle\hat{\Pi}^*\hat{\Pi}\rangle_{\mathrm{T}} = \mathrm{Tr}\Big[\hat{\rho}(t)\hat{\Pi}^*\hat{\Pi}\Big] &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\Big[\hbar^2\dot{\varphi}^*_{\mathbf{k}}(t)\dot{\varphi}_{\mathbf{k}}(t)\coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2})\Big]. \end{aligned} \tag{6.17}$$

One then finds the two-point correlation functions at equal times by taking the expectation value with respect to the vacuum state

$$G_{\mathrm{V}}(\mathbf{y},\mathbf{x},t) = \langle\hat{\Phi}^*(\mathbf{y},t)\hat{\Phi}(\mathbf{x},t)\rangle_{\mathrm{V}} = \int \frac{d^3k}{(2\pi)^3}\Big[\hbar^2\varphi^*_{\mathbf{k}}(t)\varphi_{\mathbf{k}}(t)\Big]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \tag{6.18}$$

and with respect to the thermal state

$$G_{\mathrm{T}}(\mathbf{y},\mathbf{x},t) = \langle\hat{\Phi}^*(\mathbf{y},t)\hat{\Phi}(\mathbf{x},t)\rangle_{\mathrm{T}} = \int \frac{d^3k}{(2\pi)^3}\Big[\hbar^2\varphi^*_{\mathbf{k}}(t)\varphi_{\mathbf{k}}(t)\coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2})\Big]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \tag{6.19}$$

where

$$\omega_{i,\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2(-\infty)}. \tag{6.20}$$

A. Instantaneous Quench

The instantaneous quench model is an analytically solvable one, in which the mass changes as

$$m^2(t) = \begin{cases} m_i^2, & t < 0, \\ -m_f^2, & t > 0. \end{cases} \tag{6.21}$$

Before the quench ($t < 0$), the solution to Eq. (6.11), which also satisfies the condition (3.5), is given by

$$\varphi_{i,\mathbf{k}}(t) = \frac{1}{\sqrt{2\hbar\omega_{i,\mathbf{k}}}} e^{-i\omega_{i,\mathbf{k}} t}, \quad \omega_{i,\mathbf{k}} = \sqrt{k^2 + m_i^2}. \tag{6.22}$$

Then the two-point vacuum correlation function (6.18) becomes

$$\begin{aligned} G_{i,\mathrm{V}}(\mathbf{y}, \mathbf{x}, t) &= \int \frac{d^3k}{(2\pi)^3} \frac{\hbar}{2\sqrt{k^2 + m_i^2}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \frac{\hbar}{4\pi^2} \frac{m_i K_1(m_i|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}, \end{aligned} \tag{6.23}$$

where K_1 is the modified Bessel function. Equation (6.23) coincides with the result for a massive scalar field in Ref. [49]. Similarly, the two-point thermal correlation function is given by

$$\begin{aligned} G_{i,\mathrm{T}}(\mathbf{y}, \mathbf{x}, t) &= G_{i,\mathrm{V}}(\mathbf{y}, \mathbf{x}, t) \\ &+ \frac{\hbar}{2\pi^2} \frac{m_i}{\sqrt{|\mathbf{x}-\mathbf{y}|^2 + m_i^2}} \sum_{n=1}^{\infty} K_1(m_i \sqrt{|\mathbf{x}-\mathbf{y}|^2 + (\beta\hbar n)^2})). \end{aligned} \tag{6.24}$$

On the other hand, after the quench ($t > 0$), the classical equations of motion are classified into two types: the one from the long wavelength modes with $k^2 < m_f^2$ has the negative frequency squared and exhibits an exponential behavior, and the other from the short wavelength modes with $k^2 > m_f^2$ still has the positive frequency squared and shows an oscillatory behavior. Each mode moves under a constant frequency squared before the quench time but suddenly experiences a potential step in the case of short wavelengths and a potential barrier in the case of long wavelengths. There is the analogy between Eq. (6.11) and the scattering problem of quantum mechanics. The solution to Eq. (6.11) together with the initial asymptotic data (6.22) is the complex conjugate of the scattering wave function by either the potential step or barrier [36]. The solution to Eq. (6.11) after the quench should match at the quench time $t = 0$ continuously with Eq. (6.22) before the quench. It is rather straightforward to find such solutions for the short wavelength modes ($k^2 > m_f^2$)

$$\varphi_{fS,\mathbf{k}}(t) = \frac{1}{\sqrt{2\hbar\omega_{i,\mathbf{k}}}} \left[-i\frac{\omega_{i,\mathbf{k}}}{\omega_{f,\mathbf{k}}} \sin(\omega_{f,\mathbf{k}} t) + \cos(\omega_{f,\mathbf{k}} t) \right], \quad \omega_{f,\mathbf{k}} = \sqrt{k^2 - m_f^2}, \tag{6.25}$$

and for the long wavelength modes ($k^2 < m_f^2$)

$$\varphi_{f_U,\mathbf{k}}(t) = \frac{1}{\sqrt{2\hbar\omega_{i,\mathbf{k}}}}\Big[-i\frac{\omega_{i,\mathbf{k}}}{\tilde{\omega}_{f,\mathbf{k}}}\sinh(\tilde{\omega}_{f,\mathbf{k}}t) + \cosh(\tilde{\omega}_{f,\mathbf{k}}t)\Big], \quad \tilde{\omega}_{f,\mathbf{k}} = \sqrt{m_f^2 - k^2}. \tag{6.26}$$

A few comments are in order. The solution (6.26) represents an instability due to the phase transition and is obtained by continuing analytically the solution (6.25). When Eq. (6.25) is rewritten as

$$\varphi_{f,\mathbf{k}}(t) = \frac{1}{\sqrt{2\hbar\omega_{i,\mathbf{k}}}}\Big[\Big(\frac{\omega_{f,\mathbf{k}} + \omega_{i,\mathbf{k}}}{2\omega_{f,\mathbf{k}}}\Big)e^{-i\omega_{f,\mathbf{k}}t} + \Big(\frac{\omega_{f,\mathbf{k}} - \omega_{i,\mathbf{k}}}{2\omega_{f,\mathbf{k}}}\Big)e^{i\omega_{f,\mathbf{k}}t}\Big], \tag{6.27}$$

the first and second terms correspond to the positive and negative frequencies, respectively, hence the second term explains the particle creation by changing frequency [35].

After some manipulations of algebra, we obtain the two-point vacuum correlation function after the quench

$$\begin{aligned} G_{f,\mathrm{V}}(\mathbf{y},\mathbf{x},t) &= G_{i,\mathrm{V}}(\mathbf{y},\mathbf{x},t) \\ &+ \frac{\hbar}{4\pi^2|\mathbf{x}-\mathbf{y}|}\int_0^{m_f} dk k\Big(\frac{\omega_{i,\mathbf{k}}^2 + \tilde{\omega}_{f,\mathbf{k}}^2}{\omega_{i,\mathbf{k}}\tilde{\omega}_{f,\mathbf{k}}^2}\Big)\sin(k|\mathbf{x}-\mathbf{y}|)\sinh^2(\tilde{\omega}_{f,\mathbf{k}}t) \\ &+ \frac{\hbar}{4\pi^2|\mathbf{x}-\mathbf{y}|}\int_{m_f}^{\infty} dk k\Big(\frac{\omega_{i,\mathbf{k}}^2 - \omega_{f,\mathbf{k}}^2}{\omega_{i,\mathbf{k}}\omega_{f,\mathbf{k}}^2}\Big)\sin(k|\mathbf{x}-\mathbf{y}|)\sin^2(\omega_{f,\mathbf{k}}t). \end{aligned} \tag{6.28}$$

Similarly, the two-point thermal correlation function is given by

$$\begin{aligned} G_{f,\mathrm{T}}(\mathbf{y},\mathbf{x},t) &= G_{i,\mathrm{T}}(\mathbf{y},\mathbf{x},t) \\ &+ \frac{\hbar}{4\pi^2|\mathbf{x}-\mathbf{y}|}\int_0^{m_f} dk k\Big(\frac{\omega_{i,\mathbf{k}}^2 + \tilde{\omega}_{f,\mathbf{k}}^2}{\omega_{i,\mathbf{k}}\tilde{\omega}_{f,\mathbf{k}}^2}\Big)\sin(k|\mathbf{x}-\mathbf{y}|)\sinh^2(\tilde{\omega}_{f,\mathbf{k}}t)\coth\Big(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}\Big) \\ &+ \frac{\hbar}{4\pi^2|\mathbf{x}-\mathbf{y}|}\int_{m_f}^{\infty} dk k\Big(\frac{\omega_{i,\mathbf{k}}^2 - \omega_{f,\mathbf{k}}^2}{\omega_{i,\mathbf{k}}\omega_{f,\mathbf{k}}^2}\Big)\sin(k|\mathbf{x}-\mathbf{y}|)\sin^2(\omega_{f,\mathbf{k}}t)\coth\Big(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}\Big). \end{aligned} \tag{6.29}$$

The first terms in Eqs. (6.28) and (6.29) are the two-point vacuum and thermal correlations (6.23) and (6.24), respectively, before the quench. Therefore the remaining two terms describe the effect of the quench. In particular, the second terms are dominant and rooted on the instability during the phase transition, which is missing in the field theoretical approach to equilibrium dynamics. Note that $\omega_{i,\mathbf{k}}^2 - \omega_{f,\mathbf{k}}^2 = m_i^2 + m_f^2$ and $\omega_{i,\mathbf{k}}^2 + \tilde{\omega}_{f,\mathbf{k}}^2 = m_i^2 + m_f^2$, so the amplitudes of $\sin(k|\mathbf{x}-\mathbf{y}|)$ decrease as $1/k^3$ for very short wavelengths, hence short wavelengths contribute negligibly. However, there is a residual contribution from near the critical wavelength $k_c = m_f$, which becomes much smaller than the second terms at later times and will not be considered any more.

We wish to determine the size of domains from the second order phase transition of the instantaneous quench. At later times ($m_f t \gg 1$) after the quench the dominant contribution to Eq. (6.29) comes from the second term, so one has approximately

$$G_{f_U,\mathrm{T}}(r,t) \simeq \frac{\hbar}{16\pi^2 r}\int_0^{m_f} dk\Big\{ke^{2\tilde{\omega}_{f,\mathbf{k}}t}\Big\}\sin(rk)F_{\mathrm{I}}(k), \tag{6.30}$$

where $r = |\mathbf{x}-\mathbf{y}|$, and

$$F_{\rm I}(k) = \left(\frac{m_i^2 + m_f^2}{\omega_{i,\mathbf{k}}\tilde{\omega}_{f,\mathbf{k}}^2}\right) \coth\Bigl(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}\Bigr). \tag{6.31}$$

The function of k in the curly bracket of Eq. (6.30) has a sharp peak at $k_0 = \sqrt{m_f/2t}$, whereas $F_{\rm I}(k)$ is a slowly varying function. We employ the steepest descent method (see Appendix C) to obtain

$$G_{f_U,\rm T}(r,t) \simeq G_{f_U,\rm T}(0,t) \frac{\sin\Bigl(\sqrt{\frac{m_f}{2t}}r\Bigr)}{\sqrt{\frac{m_f}{2t}}r} \exp\Bigl[-\frac{m_f r^2}{8t}\Bigr], \tag{6.32}$$

where

$$G_{f_U,\rm T}(0,t) = \frac{\hbar}{16\pi^2}\sqrt{\frac{\pi}{2e}}\Bigl(\frac{m_f}{2t}\Bigr)^{3/2} e^{2m_f t} F_{\rm I}\Bigl(k_0 = \sqrt{\frac{m_f}{2t}}\Bigr). \tag{6.33}$$

Therefore the size of domains grows according to the classical Cahn-Allen scaling relation [3]

$$\xi_D(t) = \sqrt{\frac{8t}{m_f}}. \tag{6.34}$$

The scaling relation (6.34) for the instantaneous quench confirms the result obtained in Refs. [12,13].

B. Finite Smooth Quench

The instantaneous quench does not exhibit the essential spinodal behavior during the quench process. To see the dynamics of the second order phase transition one needs a finite quench period. Such a finite and smooth quench model is described by a field with the mass given by

$$m^2(t) = -m_1^2 - m_0^2 \tanh\Bigl(\frac{t}{\tau}\Bigr), \quad (m_0^2 > |m_1^2|). \tag{6.35}$$

At earlier times $t = -\infty$, the mass has the initial value

$$m^2 = m_i^2 = m_0^2 - m_1^2 > 0, \tag{6.36}$$

and at later times $t = \infty$, the final value

$$m^2 = -m_f^2 = -(m_0^2 + m_1^2) < 0. \tag{6.37}$$

Here τ measures the quench rate: the large τ-limit implies that the mass changes slowly from m_i^2 at $t = -\infty$ to $-m_f^2$ at $t = +\infty$, whereas the small τ-limit implies a rapid change of the mass. In particular, the ($\tau = 0$)-limit corresponds to the instantaneous change from m_i^2 to $-m_f^2$ at $t = 0$. That is, the instantaneous quench is a special case of the finite smooth quench model. To find the Fock space for each mode one needs to solve the classical equation of motion

$$\ddot{\varphi}_{\mathbf{k}}(t) + \Big(\mathbf{k}^2 - m_1^2 - m_0^2 \tanh\Big(\frac{t}{\tau}\Big)\Big)\varphi_{\mathbf{k}}(t) = 0. \tag{6.38}$$

It should be noted that, as in the instantaneous quench model, long wavelength modes ($k \leq k_c = m_f$) let the frequency change the sign at later times ($t \gg \tau$)

$$\omega_{\mathbf{k}}^2(t) = \mathbf{k}^2 - m_1^2 - m_0^2 < 0, \tag{6.39}$$

and suffer from the spinodal instability. Each long wavelength mode has a different quench time determined by $\omega_{\mathbf{k}}(t_{\mathbf{k}}) = 0$.

The solutions to Eq. (6.38) are found separately for the stable modes and unstable modes. The stable modes ($k \geq m_f$) have the solutions

$$\varphi_{\mathbf{k}}(t) = C_{\mathbf{k}} e^{2p_{\mathbf{k}}t} {}_2F_1(\beta_{+,\mathbf{k}}, \beta_{-,\mathbf{k}}; \gamma_{\mathbf{k}}; -e^{2t/\tau}), \tag{6.40}$$

where

$$\begin{aligned} p_{\mathbf{k}} &= -i\frac{1}{2}\omega_{i,\mathbf{k}}, \\ \beta_{\pm,\mathbf{k}} &= -i\frac{\tau}{2}\Big(\omega_{i,\mathbf{k}} \pm \omega_{f,\mathbf{k}}\Big), \\ \gamma_{\mathbf{k}} &= 1 - i\tau\omega_{i,\mathbf{k}}, \end{aligned} \tag{6.41}$$

with

$$\omega_{i,\mathbf{k}} = \sqrt{k^2 + m_i^2}, \quad \omega_{f,\mathbf{k}} = \sqrt{k^2 - m_f^2}. \tag{6.42}$$

Whereas the unstable modes ($k < m_f$) have the solutions

$$\varphi_{\mathbf{k}}(t) = C_{\mathbf{k}} e^{2p_{\mathbf{k}}t} {}_2F_1(\tilde{\beta}_{+,\mathbf{k}}, \tilde{\beta}_{-,\mathbf{k}}; \gamma_{\mathbf{k}}; -e^{2t/\tau}), \tag{6.43}$$

where

$$\tilde{\beta}_{\pm,\mathbf{k}} = -\frac{\tau}{2}\Big(i\omega_{i,\mathbf{k}} \pm \tilde{\omega}_{f,\mathbf{k}}\Big), \tag{6.44}$$

with

$$\tilde{\omega}_{f,\mathbf{k}} = \sqrt{m_f^2 - k^2}. \tag{6.45}$$

At earlier times ($\tau \ll -\tau$) before the quench begins, both the solutions (6.40) and (6.43) have the same asymptotic form

$$\varphi_{i,\mathbf{k}}(t) = C_{\mathbf{k}} e^{-i\omega_{i,\mathbf{k}}t}, \tag{6.46}$$

so the constant is fixed to satisfy Eq. (3.5)

$$c_{\mathbf{k}} = \frac{1}{\sqrt{2\hbar\omega_{i,\mathbf{k}}}}. \tag{6.47}$$

1. During the Quench

During the quench process ($|t| < \tau$), the asymptotic forms for the solutions (6.40) and (6.43) are unfortunately not available. Instead, one may expand the mass (6.35) to the linear order

$$m^2(t) = -m_1^2 - m_0^2\Big(\frac{t}{\tau}\Big), \tag{6.48}$$

which is a good approximation as far as $|t| \ll \tau$. But we assume $m_0\tau \gg 1$ so that the linear quench process is sufficiently long enough to allow the asymptotic analysis. Each mode of the scalar field then has the frequency squared

$$\omega_{\mathbf{k}}^2(t) = \mathbf{k}^2 - m_1^2 - m_0^2\Big(\frac{t}{\tau}\Big) = m_0^2\left(\frac{t_{\mathbf{k}} - t}{\tau}\right), \tag{6.49}$$

where $t_{\mathbf{k}}$ is the quench time for the corresponding unstable mode

$$t_{\mathbf{k}} = \frac{\tau}{m_0^2}(\mathbf{k}^2 - m_1^2). \tag{6.50}$$

Unless $|t_{\mathbf{k}}/\tau| \ll 1$, the quench time $t_{\mathbf{k}}$ occurs outside the valid regime for Eq. (6.48). So we restrict our attention to those unstable modes with $t_{\mathbf{k}} \ll \tau$. The frequency (6.49) corresponds to the special case of Eq. (4.17), in which $l_1 = l_2 = 0$, $t_0 = t_{\mathbf{k}}$, $t_0 - t_i = \tau$ and $\omega_i = m_0$. Then the solution in the intermediate regime before the quench, matching with the initial solution (6.46), is given by Eq. (4.18):

$$\varphi_{m_S,\mathbf{k}}(t) = D_1 z_{\mathbf{k}}^{1/3} H_{1/3}^{(2)}(z_{\mathbf{k}}) + D_2 z_{\mathbf{k}}^{1/3} H_{1/3}^{(1)}(z_{\mathbf{k}}), \tag{6.51}$$

where

$$z_{\mathbf{k}} = \frac{2}{3} m_0 \tau \left(\frac{t_{\mathbf{k}} - t}{\tau}\right)^{1/3}. \tag{6.52}$$

The coefficients D_1 and D_2 are given by Eq. (4.20). After the quench time ($t > t_{\mathbf{k}}$) for each mode, the solution (6.51) is analytically continued for the unstable mode

$$\begin{aligned}\varphi_{m_U,\mathbf{k}}(t) = \frac{D_1}{2}\tilde{z}_{\mathbf{k}}^{1/3}&\Big[e^{i\frac{3}{2}\pi} H_{1/3}^{(2)}(i\tilde{z}_{\mathbf{k}}) + e^{i\frac{1}{2}\pi} H_{1/3}^{(2)}(-i\tilde{z}_{\mathbf{k}})\Big] \\ +\frac{D_2}{2}\tilde{z}_{\mathbf{k}}^{1/3}&\Big[e^{i\frac{3}{2}\pi} H_{1/3}^{(1)}(i\tilde{z}_{\mathbf{k}}) + e^{i\frac{1}{2}\pi} H_{1/3}^{(1)}(-i\tilde{z}_{\mathbf{k}})\Big],\end{aligned} \tag{6.53}$$

where

$$\tilde{z}_{\mathbf{k}} = \frac{2}{3} m_0 \tau \left(\frac{t - t_{\mathbf{k}}}{\tau}\right)^{1/3}. \tag{6.54}$$

The two-point thermal correlation function

$$G_{m,\mathrm{T}}(r,t) = \frac{\hbar^2}{2\pi^2}\int_0^{m_f} dk k^2 \frac{\sin(kr)}{kr}\varphi^*_{m_U,\mathbf{k}}(t)\varphi_{m_U,\mathbf{k}}(t) + \frac{\hbar^2}{2\pi^2}\int_{m_f}^{\infty} dk k^2 \frac{\sin(kr)}{kr}\varphi^*_{m_S,\mathbf{k}}(t)\varphi_{m_S,\mathbf{k}}(t), \tag{6.55}$$

with $r = |\mathbf{x}-\mathbf{y}|$, is dominated by the unstable modes (6.53), since the stable modes (6.51) oscillate rapidly and do contribute little. By using the asymptotic form for the solution (6.53) in the regime $\tau \gg t \gg (\tau/m_0^2)^{1/3}$

$$\varphi_{m_U,\mathbf{k}}(t) = \sqrt{\frac{1}{2\pi}}\frac{e^{\tilde{z}_\mathbf{k}}}{\tilde{z}_\mathbf{k}^{1/6}}\Big[e^{-i\frac{1}{3}\pi}D_1 + e^{-i\frac{2}{3}\pi}D_2\Big]. \tag{6.56}$$

one has approximately

$$\varphi^*_{m_U,\mathbf{k}}(t)\varphi_{m_U,\mathbf{k}}(t) = \frac{1}{8\hbar}\Big\{\frac{1}{m_0}\Big(\frac{z_i}{\tilde{z}_\mathbf{k}}\Big)^{1/3}\sin^2\Big(z_i + \frac{\pi}{4}\Big)\Big\}e^{2\tilde{z}_\mathbf{k}}, \tag{6.57}$$

where $z_i = 2m_0\tau/3$. So the correlation function takes the form

$$G_{m_U,\mathrm{T}}(r,t) \simeq \frac{\hbar}{16\pi^2 r}\int_0^{m_f} dk\Big\{k e^{2\tilde{\omega}_{f,\mathbf{k}}\tilde{t}}\Big\}\sin(kr)F_{\mathrm{II}}(k), \tag{6.58}$$

where

$$F_{\mathrm{II}}(k) = \frac{1}{m_0}\Big(\frac{\frac{2}{3}m_0\tau}{\tilde{z}_\mathbf{k}}\Big)^{1/3}\sin^2\Big(\frac{2}{3}m_0\tau + \frac{\pi}{4}\Big)\coth\Big(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}\Big). \tag{6.59}$$

The function in the curly bracket is rapidly varying and has a peak at $k_0 = (m_0/2\sqrt{\tau t})^{1/2}$. Hence after applying the steepest decent method (see Appendix C), we finally obtain

$$G_{m_U,\mathrm{T}}(r,t) \simeq G_{m_U,\mathrm{T}}(0,t)\frac{\sin\Big(\frac{\sqrt{\tau t}}{m_0}r\Big)}{\frac{\sqrt{\tau t}}{m_0}r}\exp\Big[-\frac{r^2}{8\frac{\sqrt{\tau t}}{m_0}}\Big], \tag{6.60}$$

where

$$G_{m_U,\mathrm{T}}(0,t) = \frac{\hbar}{64\pi^2 m_0}\left(\frac{\pi m_0^3}{(\tau t)^{3/2}}\right)^{1/2} e^{(\frac{4}{3}m_0 t + 2\frac{m_1^2}{m_0}\tau)\sqrt{\frac{t}{\tau}}}F_{II}\Big(k_0 = (\frac{m_0}{2\sqrt{\tau t}})^{1/2}\Big). \tag{6.61}$$

We thus have shown the scaling relation for the domain size

$$\xi_D(t) = 2\left(\frac{2\tau t}{m_0^2}\right)^{1/4}. \tag{6.62}$$

The scaling relation (6.62) confirms, up to a numerical factor, the result in Ref. [16]. However, the power-law is different from the Cahn-Allen scaling relation after the completion of quench.

At later times ($t \gg \tau$) after the completion of quench, the solution (6.40) has the asymptotic form

$$\varphi_{f,\mathbf{k}}(t) = \frac{e^{2p_{\mathbf{k}}t}}{\sqrt{2\hbar\omega_{i,\mathbf{k}}}}\Big[\frac{\Gamma(\gamma_{\mathbf{k}})\Gamma(\beta_{-,\mathbf{k}}-\beta_{+,\mathbf{k}})}{\Gamma(\beta_{-,\mathbf{k}})\Gamma(\gamma_{\mathbf{k}}-\beta_{+,\mathbf{k}})}e^{-2\beta_{+,\mathbf{k}}t/\tau}{}_2F_1(\beta_{+,\mathbf{k}}, 1-\gamma_{\mathbf{k}}+\beta_{+,\mathbf{k}}; 1-\beta_{-,\mathbf{k}}+\beta_{+,\mathbf{k}}; -e^{-2t/\tau})$$
$$+\frac{\Gamma(\gamma_{\mathbf{k}})\Gamma(\beta_{+,\mathbf{k}}-\beta_{-,\mathbf{k}})}{\Gamma(\beta_{+,\mathbf{k}})\Gamma(\gamma_{\mathbf{k}}-\beta_{-,\mathbf{k}})}e^{-2\beta_{-,\mathbf{k}}t/\tau}{}_2F_1(\beta_{-,\mathbf{k}}, 1-\gamma_{\mathbf{k}}+\beta_{-,\mathbf{k}}; 1-\beta_{+,\mathbf{k}}+\beta_{-,\mathbf{k}}; -e^{-2t/\tau})\Big]. \quad (6.63)$$

The asymptotic form for Eq. (6.43) is obtained by replacing $\beta_{\pm,\mathbf{k}}$ by $\tilde{\beta}_{\pm,\mathbf{k}}$. From the asymptotic form of the hypergeometric function [42], we find the asymptotic form for the stable modes

$$\varphi_{f_S,\mathbf{k}}(t) = \frac{1}{\sqrt{2\hbar\omega_{i,\mathbf{k}}}}\frac{\Gamma(1-i\omega_{i,\mathbf{k}}\tau)\Gamma(-i\omega_{f,\mathbf{k}}\tau)}{\Gamma(1-i\frac{\tau}{2}(\omega_{i,\mathbf{k}}+\omega_{f,\mathbf{k}}))\Gamma(-i\frac{\tau}{2}(\omega_{i,\mathbf{k}}+\omega_{f,\mathbf{k}}))}e^{-i\omega_{f,\mathbf{k}}t}$$
$$+\frac{1}{\sqrt{2\hbar\omega_{i,\mathbf{k}}}}\frac{\Gamma(1-i\omega_{i,\mathbf{k}}\tau)\Gamma(i\omega_{f,\mathbf{k}}\tau)}{\Gamma(1-i\frac{\tau}{2}(\omega_{i,\mathbf{k}}-\omega_{f,\mathbf{k}}))\Gamma(-i\frac{\tau}{2}(\omega_{i,\mathbf{k}}-\omega_{f,\mathbf{k}}))}e^{i\omega_{f,\mathbf{k}}t}, \quad (6.64)$$

and for the unstable modes

$$\varphi_{f_U,\mathbf{k}}(t) = \frac{1}{\sqrt{2\hbar\omega_{i,\mathbf{k}}}}\frac{\Gamma(1-i\omega_{i,\mathbf{k}}\tau)\Gamma(\tilde{\omega}_{f,\mathbf{k}}\tau)}{(-1)\frac{\tau}{2}(i\omega_{i,\mathbf{k}}-\tilde{\omega}_{f,\mathbf{k}})\Gamma^2(-\frac{\tau}{2}(i\omega_{i,\mathbf{k}}-\tilde{\omega}_{f,\mathbf{k}}))}e^{\tilde{\omega}_{f,\mathbf{k}}t}$$
$$+\frac{1}{\sqrt{2\hbar\omega_{i,\mathbf{k}}}}\frac{\Gamma(1-i\omega_{i,\mathbf{k}}\tau)\Gamma(-\tilde{\omega}_{f,\mathbf{k}}\tau)}{(-1)\frac{\tau}{2}(i\omega_{i,\mathbf{k}}+\tilde{\omega}_{f,\mathbf{k}})\Gamma^2(-\frac{\tau}{2}(i\omega_{i,\mathbf{k}}+\tilde{\omega}_{f,\mathbf{k}}))}e^{-\tilde{\omega}_{f,\mathbf{k}}t}. \quad (6.65)$$

Now a few comments are in order. First, the coefficient of the positive frequency asymptotic solution for each short wavelength mode

$$\varphi^{out}_{\mathbf{k}}(t) = \frac{1}{\sqrt{2\hbar\omega_{f,\mathbf{k}}}}e^{-i\omega_{f,\mathbf{k}}t} \quad (6.66)$$

leads to the rate for the initial vacuum to remain in the final vacuum

$$|\langle 0_{\mathbf{k}}, +\infty|0_{\mathbf{k}}, -\infty\rangle|^2 = \frac{\sinh^2\Big[\frac{\pi\tau}{2}\Big(\omega_{i,\mathbf{k}}+\omega_{f,\mathbf{k}}\Big)\Big]}{\sinh(\pi\tau\omega_{i,\mathbf{k}})\sinh(\pi\tau\omega_{f,\mathbf{k}})}. \quad (6.67)$$

On the other hand, the coefficient of the negative frequency solution $\varphi^{out*}_{\mathbf{k}}(t)$ leads to the particle production rate [35]

$$1-|\langle 0_{\mathbf{k}}, +\infty|0_{\mathbf{k}}, -\infty\rangle|^2 = \frac{\sinh^2\Big[\frac{\pi\tau}{2}\Big(\omega_{i,\mathbf{k}}-\omega_{f,\mathbf{k}}\Big)\Big]}{\sinh(\pi\tau\omega_{i,\mathbf{k}})\sinh(\pi\tau\omega_{f,\mathbf{k}})}. \quad (6.68)$$

Second, when $\tilde{\omega}_{f\mathbf{k}}\tau \geq 1$, the coefficient of the decaying mode in Eq. (6.65) can become infinite at

$$\tilde{\omega}_{f,\mathbf{k}}\tau = n, \quad (n = 1, 2, 3, \cdots), \tag{6.69}$$

because the gamma function has simple poles at these negative integers. For a rapid quench $\tau \ll 1$, there does not exist any integers that satisfy Eq. (6.69). Hence this kind of resonance can happen only for a non-negligible τ, *i.e.*, for a very slow quench, which will be treated elsewhere. Contrary to the resonance at (6.69), the apparent singularity at $\tilde{\omega}_{f,\mathbf{k}}\tau = 0$ is removed by considering both terms in Eq. (6.65):

$$\begin{aligned}\varphi_{f_U\mathbf{k}}(t) &= \frac{1}{\sqrt{2\hbar\omega_{i,\mathbf{k}}}} \frac{\Gamma(1 - i\omega_{i,\mathbf{k}}\tau)}{(-i\frac{\tau}{2}\omega_{i,\mathbf{k}})\Gamma^2(-i\frac{\tau}{2}\omega_{i,\mathbf{k}})} \Big[\Gamma(-\tilde{\omega}_{f,\mathbf{k}}\tau)e^{-\tilde{\omega}_{f,\mathbf{k}}t} + \Gamma(\tilde{\omega}_{f,\mathbf{k}}\tau)e^{\tilde{\omega}_{f,\mathbf{k}}t}\Big] \\ &= \frac{1}{\sqrt{2\hbar\omega_{i,\mathbf{k}}}} \frac{\Gamma(1 - i\omega_{i,\mathbf{k}}\tau)}{(-i\frac{\tau}{2}\omega_{i,\mathbf{k}})\Gamma^2(-i\frac{\tau}{2}\omega_{i,\mathbf{k}})} \Big[2\frac{\sinh(\tilde{\omega}_{f,\mathbf{k}}t)}{\tilde{\omega}_{f,\mathbf{k}}\tau}\Big].\end{aligned} \tag{6.70}$$

After the completion of quench ($t \gg \tau$), the unstable modes (6.65) dominate the correlation function (6.55) over the stable modes (6.64). In particular, the first term of Eq. (6.65) grows exponentially, so by using the asymptotic form in Appendix D, one has approximately

$$\varphi^*_{f_U,\mathbf{k}}(t)\varphi_{f_U,\mathbf{k}}(t) = \frac{1}{8\hbar}\Bigg\{\bigg(\frac{\omega^2_{i,\mathbf{k}} + \tilde{\omega}^2_{f,\mathbf{k}}}{\omega_{i,\mathbf{k}}\tilde{\omega}_{f,\mathbf{k}}}\bigg)\frac{\pi\omega_{i,\mathbf{k}}\tau}{\sinh(\pi\omega_{i,\mathbf{k}})}\Bigg[\frac{\Big(1+\frac{\tau}{2}\tilde{\omega}_{f,\mathbf{k}}\Big)^2 + \frac{\tau^2}{4}\omega_{i,\mathbf{k}}}{1+\tau\tilde{\omega}_{f,\mathbf{k}}}\Bigg]^2 e^{\frac{\tau^2}{2}(\zeta(2)-1)(\omega^2_{i,\mathbf{k}}+\tilde{\omega}^2_{f,\mathbf{k}})}\Bigg\}e^{2\tilde{\omega}_{f,\mathbf{k}}\tilde{t}}, \tag{6.71}$$

where $\zeta(n)$ is the Riemann zeta function and

$$\tilde{t} = t - \frac{\tau^3}{8}\Big(\zeta(3) - 1\Big)\Big(\omega^2_{i,\mathbf{k}} + \tilde{\omega}^2_{f,\mathbf{k}}\Big). \tag{6.72}$$

Note that $\omega^2_{i,\mathbf{k}} + \tilde{\omega}^2_{f,\mathbf{k}} = m_i^2 + m_f^2$, so $\tilde{t}$ lags by a constant in proportion to the cubic power of the quench period τ. This time-lag is determined not only by the quench period but also by the initial and final coupling constants m_i and m_f. After applying the steepest decent method to the correlation function (see Appendix C), we finally obtain

$$G_{f_U,\mathrm{T}}(r,t) \simeq G_{f_U,\mathrm{T}}(0,t)\frac{\sin\Big(\sqrt{\frac{m_f}{2\tilde{t}}}r\Big)}{\sqrt{\frac{m_f}{2\tilde{t}}}r}\exp\Big[-\frac{m_f r^2}{8\tilde{t}}\Big], \tag{6.73}$$

where

$$G_{f_U,\mathrm{T}}(0,t) = \frac{\hbar}{16\pi^2}\sqrt{\frac{\pi}{2e}}\Big(\frac{m_f}{2\tilde{t}}\Big)^{3/2}e^{2m_f\tilde{t}}F_{\mathrm{III}}\Big(k_0 = \sqrt{\frac{m_f}{2\tilde{t}}}\Big). \tag{6.74}$$

Here

$$F_{\mathrm{III}}(k) = \bigg(\frac{\omega^2_{i,\mathbf{k}} + \tilde{\omega}^2_{f,\mathbf{k}}}{\omega_{i,\mathbf{k}}\tilde{\omega}_{f,\mathbf{k}}}\bigg)\frac{\pi\omega_{i,\mathbf{k}}\tau}{\sinh(\pi\omega_{i,\mathbf{k}})}\Bigg[\frac{\Big(1+\frac{\tau}{2}\tilde{\omega}_{f,\mathbf{k}}\Big)^2 + \frac{\tau^2}{4}\omega_{i,\mathbf{k}}}{1+\tau\tilde{\omega}_{f,\mathbf{k}}}\Bigg]^2 e^{\frac{\tau^2}{2}(\zeta(2)-1)(\omega^2_{i,\mathbf{k}}+\tilde{\omega}^2_{f,\mathbf{k}})}\coth\Big(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}\Big). \tag{6.75}$$

Remarkably, the scaling relation of the domain size has the same form as Eq. (6.34) from the instantaneous quench

$$\xi_D(t) = \sqrt{\frac{8\tilde{t}}{m_f}}. \tag{6.76}$$

However, there is a definite time-lag due to the finite quench as claimed in Ref. [13]. The scaling relation (6.76) is robust, because only the asymptotic form of the exact solutions (6.43) is used.

VII. BACK-REACTION IN $U(1)$-THEORY FOR PHASE TRANSITIONS

The free scalar field model in Sec. VI does not describe a real system for the second order phase transition because the spinodal instability continues indefinitely. This model describes more appropriately an intermediate process of phase transition toward the spinodal line. In this section we consider $|\Phi|^4$-theory, in which there is a natural exit for the spinodal decomposition. The spinodal decomposition ends at the spinodal line, the regime in which the instability from the quench competes with the back-reaction from the $|\Phi|^4$-potential term.

The $U(1)$-model for the second order phase transition is described by the Lagrangian density

$$\mathcal{L}(t) = \dot{\Phi}^*\dot{\Phi} - \nabla\Phi^* \cdot \nabla\Phi - \frac{\lambda}{4!}\left(\Phi^*\Phi + \frac{12m^2(t)}{\lambda}\right)^2, \tag{7.1}$$

where $m^2(t)$ is assumed to take either (6.21) or (6.35). Hence, the coupling parameter $m^2(t)$ of the quadratic term starts with the positive constant value m_i^2 at earlier times before the quench, keeps decreasing across zero at the moment of quench t_0, and finally reaches the negative constant value $-m_f^2$ at later times after the quench. So the $\Phi = 0$ remains the true minimum of the potential until t_0. However, $\Phi = 0$ is no longer the true minimum after t_0, but becomes a local maximum. The true minimum occurs at $|\Phi|^2 = 12m^2(t)/\lambda$, and the system undergoes a second order phase transition. When the system is in the initial thermal equilibrium at earlier times, it is invariant under $\Phi \to \Phi e^{i\theta}$ and has a global $U(1)$-symmetry. As the system undergoes the second order phase transition, the symmetry is broken and there occurs topological defects.

¿From the Lagrangian density (7.1) follows the Hamiltonian density

$$\mathcal{H}(\mathbf{x}, t) = \Pi^*\Pi + \nabla\Phi^* \cdot \nabla\Phi^* + m^2(t)\Phi^*\Phi + \frac{\lambda}{4!}(\Phi^*\Phi)^2, \tag{7.2}$$

where we neglected a time-dependent c-number term. Since it has been shown in Sec. V that the coherent and coherent-thermal states in the LvN approach give the identical result as the mean-field and Hartree-Fock methods, we first divide the field into a classical background and quantum fluctuations and then quantize the latter according to the LvN approach. The field may be divided into the homogeneous classical background field and quantum fluctuations

$$\Phi(\mathbf{x}, t) = \phi_c(t) + \Phi_f(\mathbf{x}, t) \tag{7.3}$$

such that the classical background is a coherent state of the quantum field and the quantum fluctuations have symmetric states such as the vacuum, number and thermal states:

$$\langle \hat{\Phi} \rangle = \phi_c(t), \quad \langle \hat{\Phi}_f \rangle = 0. \tag{7.4}$$

The Hamiltonian density is then the sum of the classical background, quantum fluctuations and interactions

$$\mathcal{H}(\mathbf{x}, t) = \mathcal{H}_c(t) + \mathcal{H}_f(\mathbf{x}, t) + \mathcal{H}_{int}(\mathbf{x}, t) + \delta\mathcal{H}_{int}(\mathbf{x}, t), \tag{7.5}$$

where

$$\begin{aligned}
\mathcal{H}_c(t) &= \pi_c^2 + m^2(t)\phi_c^2 + \frac{\lambda}{4!}\phi_c^4, \\
\mathcal{H}_f(\mathbf{x}, t) &= \Pi_f^* \Pi_f + \nabla\Phi_f^* \cdot \nabla\Phi_f + m^2(t)\Phi_f^*\Phi_f + \frac{\lambda}{4!}(\Phi_f^*\Phi_f)^2, \\
\mathcal{H}_{int}(\mathbf{x}, t) &= \frac{\lambda}{3!}\phi_c^2(\Phi_f^*\Phi_f), \\
\delta\mathcal{H}_{int}(\mathbf{x}, t) &= \pi_c(\Pi_f^* + \Pi_f) + (\nabla\phi_c)\cdot(\nabla\Phi_f^* + \nabla\Phi_f) + m^2(t)\phi_c(\Phi_f^* + \Phi_f) \\
&\quad + \frac{\lambda}{4!}\Big\{ \phi_c^2(\Phi_f^{*2} + \Phi_f^2) + 2\phi_c^3(\Phi_f^* + \Phi_f)\Big) + 2\phi_c(\Phi_f^*\Phi_f)(\Phi_f^* + \Phi_f)\Big\},
\end{aligned} \tag{7.6}$$

with $\pi_c = \dot{\phi}_c, \pi_f = \dot{\phi}_f$. As we are interested in the symmetric state of quantum fluctuations Φ_f, so the expectation value of $\delta\mathcal{H}_{int}$ vanishes and will not be considered any more, since only the terms involving $(\Phi_f^*\Phi_f)$ do not vanish when one takes the expectation value with respect to the symmetric state.

Now the field Φ_f and Φ_f^* and their conjugate momenta are decomposed into Fourier-modes according to Eqs. (6.4) and (6.5). The Fourier transform of the quartic term leads to

$$\int d^3\mathbf{x} (\Phi_f^*\Phi_f)^2 = \prod_{j=1}^{4} \int \frac{d^3\mathbf{k}_j}{(2\pi)^3} (\phi_{\mathbf{k}_1}^* \phi_{\mathbf{k}_2})(\phi_{\mathbf{k}_3}^* \phi_{\mathbf{k}_4}) \delta(\mathbf{k}_1 + \mathbf{k}_3 - \mathbf{k}_2 - \mathbf{k}_4). \tag{7.7}$$

The symmetric state further restricts the integral in Eq. (7.7) to either $(\mathbf{k}_1 = \mathbf{k}_2, \mathbf{k}_3 = \mathbf{k}_4)$ or $(\mathbf{k}_1 = \mathbf{k}_4, \mathbf{k}_2 = \mathbf{k}_3)$, so Eq. (7.7) leads to

$$\langle \int d^3\mathbf{x} (\hat{\Phi}_f^* \hat{\Phi}_f)^2 \rangle = 2\Big[\int \frac{d^3\mathbf{k}}{(2\pi)^3} \langle \hat{\phi}_{\mathbf{k}}^* \hat{\phi}_{\mathbf{k}} \rangle \Big]^2 = 2\langle \hat{\Phi}_f^* \hat{\Phi}_f \rangle^2. \tag{7.8}$$

Keeping the symmetric state in mind, we find the the Fourier transform of $\mathcal{H}_f + \mathcal{H}_{int}$

$$\begin{aligned}
\mathcal{H}_f + \mathcal{H}_{int} &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[\pi_{\mathbf{k}}^*(t)\pi_{\mathbf{k}}(t) + \Big(\mathbf{k}^2 + m^2(t) + \frac{\lambda}{3!}\phi_c^2(t)\Big)\phi_{\mathbf{k}}^*(t)\phi_{\mathbf{k}}(t) \Big] \\
&\quad + \frac{\lambda}{2\cdot 3!}\Big[\int \frac{d^3\mathbf{k}}{(2\pi)^3} \phi_{\mathbf{k}}^*(t)\phi_{\mathbf{k}}(t) \Big]^2.
\end{aligned} \tag{7.9}$$

Therefore the Hamiltonian for fluctuations is mode-decomposed into the sum of infinite number of coupled anharmonic oscillators.

We apply the method for quantum anharmonic oscillators in Sec. V to the Hamiltonian (7.9). According to the LvN approach, we first introduce two pairs of the annihilation and creation operators

$$\hat{A}_{\mathbf{k}}(t) = i\Big(\varphi^*_{\mathbf{k}}(t)\hat{\pi}^*_{\mathbf{k}} - \dot{\varphi}^*_{\mathbf{k}}(t)\hat{\phi}_{\mathbf{k}}\Big),$$
$$\hat{A}^{\dagger}_{\mathbf{k}}(t) = -i\Big(\varphi_{\mathbf{k}}(t)\hat{\pi}_{\mathbf{k}} - \dot{\varphi}_{\mathbf{k}}(t)\hat{\phi}^*_{\mathbf{k}}\Big), \tag{7.10}$$

and

$$\hat{A}^*_{\mathbf{k}}(t) = i\Big(\varphi^*_{\mathbf{k}}(t)\hat{\pi}_{\mathbf{k}} - \dot{\varphi}^*_{\mathbf{k}}(t)\hat{\phi}^*_{\mathbf{k}}\Big),$$
$$\hat{A}^{*\dagger}_{\mathbf{k}}(t) = -i\Big(\varphi_{\mathbf{k}}(t)\hat{\pi}^*_{\mathbf{k}} - \dot{\varphi}_{\mathbf{k}}(t)\hat{\phi}_{\mathbf{k}}\Big). \tag{7.11}$$

We then require the creation and annihilation operators (7.10) to satisfy the LvN equation to obtain the equation of motion

$$\ddot{\varphi}_{\mathbf{k}}(t) + \left[\mathbf{k}^2 + m^2(t) + \frac{\lambda}{3!}\phi_c^2(t) + \frac{\lambda}{3!}\int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\hbar^2\varphi^*_{\mathbf{k}_1}(t)\varphi_{\mathbf{k}_1}(t)\right]\varphi_{\mathbf{k}}(t) = 0, \tag{7.12}$$

where the expectation value is taken with respect to the vacuum state, a symmetric state, as mentioned. The LvN equations for the operators (7.11) are the complex conjugate of Eq. (7.12).

The Fock space for each mode can be found similarly according to Sec. V. We consider two symmetric states: the vacuum and thermal states. The vacuum state that is annihilated by all $\hat{A}_{\mathbf{k}}$ and $\hat{A}^*_{\mathbf{k}}$

$$\hat{A}_{\mathbf{k}}(t)|0,t\rangle = 0, \quad \hat{A}^*_{\mathbf{k}}(t)|0,t\rangle = 0, \tag{7.13}$$

leads to the expectation values

$$\langle\hat{\Phi}^*_f\hat{\Phi}_f\rangle_{\mathrm{V}} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\Big[\hbar^2\varphi^*_{\mathbf{k}}(t)\varphi_{\mathbf{k}}(t)\Big],$$
$$\langle\hat{\Pi}^*_f\hat{\Pi}_f\rangle_{\mathrm{V}} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\Big[\hbar^2\dot{\varphi}^*_{\mathbf{k}}(t)\dot{\varphi}_{\mathbf{k}}(t)\Big], \tag{7.14}$$

The initial thermal state defined by the density operator for each mode

$$\hat{\rho}_{\mathrm{T}}(t) = \prod_{\mathbf{k}}\hat{\rho}_{\mathbf{k}}(t) = \prod_{\mathbf{k}}\left\{\frac{1}{Z_{\mathbf{k}}}\exp\left[-\beta\hbar\omega_{i,\mathbf{k}}\Big(\hat{A}^{\dagger}_{\mathbf{k}}(t)\hat{A}_{\mathbf{k}}(t) + \frac{1}{2}\Big)\right]\right.$$
$$\left.\times\frac{1}{Z^*_{\mathbf{k}}}\exp\left[-\beta\hbar\omega_{i,\mathbf{k}}\Big(\hat{A}^{*\dagger}_{\mathbf{k}}(t)\hat{A}^*_{\mathbf{k}}(t) + \frac{1}{2}\Big)\right]\right\} \tag{7.15}$$

leads to the expectation values

$$\langle\hat{\Phi}^*_f\hat{\Phi}_f\rangle_{\mathrm{T}} = \mathrm{Tr}\Big[\hat{\rho}_{\mathrm{T}}(t)\hat{\Phi}^*_f\hat{\Phi}_f\Big] = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\Big[\hbar^2\varphi^*_{\mathbf{k}}(t)\varphi_{\mathbf{k}}(t)\coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2})\Big],$$
$$\langle\hat{\Pi}^*_f\hat{\Pi}_f\rangle_{\mathrm{T}} = \mathrm{Tr}\Big[\hat{\rho}_{\mathrm{T}}(t)\hat{\Phi}^*_f\hat{\Phi}_f\Big] = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\Big[\hbar^2\dot{\varphi}^*_{\mathbf{k}}(t)\dot{\varphi}_{\mathbf{k}}(t)\coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2})\Big]. \tag{7.16}$$

Then the effective Hamiltonian for the classical background with all the contributions from fluctuations in the initial thermal state is given by

$$H_C(t) \equiv \hat{\mathcal{H}}_c + \langle \hat{\mathcal{H}}_{int} \rangle_{\rm T} \\ = \pi_c^2 + \Big[m^2(t) + \frac{\lambda}{3!} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \hbar^2 \varphi^*_{\mathbf{k}}(t) \varphi_{\mathbf{k}}(t) \coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}) \Big] \phi_c^2 + \frac{\lambda}{4!} \phi_c^4, \tag{7.17}$$

and that for fluctuations by

$$H_{\rm F} \equiv \langle \hat{\mathcal{H}}_f + \hat{\mathcal{H}}_{int} \rangle_{\rm T} \\ = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[\hbar^2 \dot{\varphi}^*_{\mathbf{k}}(t) \dot{\varphi}_{\mathbf{k}}(t) + \Big(\mathbf{k}^2 + m^2(t) + \frac{\lambda}{3!} \phi_c^2(t) \Big) \hbar^2 \varphi^*_{\mathbf{k}}(t) \varphi_{\mathbf{k}}(t) \Big] \coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}) \\ + \frac{\lambda}{2 \cdot 3!} \Big[\int \frac{d^3\mathbf{k}}{(2\pi)^3} \hbar^2 \varphi^*_{\mathbf{k}}(t) \varphi_{\mathbf{k}}(t) \coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}) \Big]^2 . \tag{7.18}$$

The effective Hamiltonian from the initial vacuum state is the zero-temperature limit ($\beta \to \infty$), *i.e.*, $\coth(\beta\hbar\omega_{i,\mathbf{k}}/2) \to 1$.

On the other hand, in the effective action method of Sec. V.B, the $\varphi_{\mathbf{k}}$ is a parameter that will be determined by the Hamilton equations. By writing $\varphi_{\mathbf{k}}$ in the polar form

$$\varphi_{\mathbf{k}} = \frac{\zeta_{\mathbf{k}}(t)}{\sqrt{\hbar}} e^{-i\theta_{\mathbf{k}}}, \tag{7.19}$$

and by introducing $p_{\zeta_{\mathbf{k}}} = \dot{\zeta}_{\mathbf{k}}$, the effective Hamiltonian for the $\mathbf{k}$-mode (7.18) can be rewritten as

$$H_{\rm T,\mathbf{k}}(t) = \hbar \coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}) \Big[p_{\zeta_{\mathbf{k}}}^2 + \omega_{\mathbf{k}}^2(t) \zeta_{\mathbf{k}}^2 + \frac{1}{8\zeta_{\mathbf{k}}^2} \Big] = \coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}) H_{\rm V,\mathbf{k}}(t). \tag{7.20}$$

where

$$\omega_{\mathbf{k}}^2(t) = \mathbf{k}^2 + m^2(t) + \frac{\lambda}{3!} \phi_c^2(t) + \frac{\lambda\hbar}{3!} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \zeta_{\mathbf{k}}^2(t) \coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}). \tag{7.21}$$

The Hamilton equations are

$$\frac{d\zeta_{\mathbf{k}}}{dt} = \frac{\partial}{\partial p_{\zeta_{\mathbf{k}}}} \Big[\frac{H_{\rm T,\mathbf{k}}(t)}{\hbar \coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2})} \Big] = p_{\zeta_{\mathbf{k}}}, \\ \frac{dp_{\zeta_{\mathbf{k}}}}{dt} = -\frac{\partial}{\partial \zeta_{\mathbf{k}}} \Big[\frac{H_{\rm T,\mathbf{k}}(t)}{\hbar \coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2})} \Big] = -\omega_{\mathbf{k}}^2(t) \zeta_{\mathbf{k}} + \frac{1}{4\zeta_{\mathbf{k}}^3}. \tag{7.22}$$

These Hamilton equations are identical to Eq. (7.12), because the effective action method is equivalent to the LvN approach as shown in Sec. V.B. The Hamilton equations of motion for the classical background are given by

$$\ddot{\phi}_c(t) + \Big[m^2(t) + \frac{\lambda}{3!} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \hbar^2 \varphi^*_{\mathbf{k}}(t) \varphi_{\mathbf{k}}(t) \coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}) \Big] \phi_c(t) + \frac{\lambda}{2 \cdot 3!} \phi_c^3 = 0. \tag{7.23}$$

The different quantum states given in Sec. V can be chosen to describe the different processes for the phase transition. We assume that the system starts from either the thermal or coherent-thermal states before the quench and evolves according to the functional Schrödinger equation during and after the quench. In this sense the process is completely determined once the initial quantum state is prescribed. In the first case of the symmetric thermal state, each mode has the zero expectation value for the position and momentum, but its dynamics is still governed by Eq. (7.12) with $\phi_c = 0$. In the second case of the coherent-thermal state, the classical field ϕ_c, which is a coherent state of the homogeneous field, plays the role of an order parameter and is influenced by thermal fluctuations of $\phi_{\mathbf{k}}$. As the first case is a limit of the second one, we shall first focus on the coherent-thermal and then obtain the thermal state result by taking the limit $\phi_c(t) = \dot{\phi}_c(t) = 0$ in the end.

It is very difficult to solve analytically the equations of motion (7.12) and (7.23). At best we have to rely on the adiabatic solutions that can be found in some important physical regimes. We assume that far before the quench ($t \to -\infty$) the system starts from a thermal fluctuations around the order parameter with

$$\phi_c(-\infty) \approx 0, \quad \dot{\phi}_c(-\infty) \approx 0. \tag{7.24}$$

As ϕ_c is a classical field, the uncertainty principle does not prohibit us from even taking the limit $\phi_c(-\infty) = \dot{\phi}_c(-\infty) = 0$, which leads to the symmetric thermal state. The initial data for ϕ_c take very small values with high probability by some distribution function. For the sake of simplicity we consider the instantaneous quench first. Before the quench time, $m^2(t)$ takes the initial constant value m_i^2 and ϕ_c remains close to zero, so $\omega_{\mathbf{k}}$ changes very little during the evolution. The solutions to Eq. (7.12) are approximately given by

$$\varphi_{i,\mathbf{k}}(t) = \frac{1}{\sqrt{2\hbar\Omega_{i,\mathbf{k}}}} e^{-i\Omega_{i,\mathbf{k}} t}, \tag{7.25}$$

and Eq. (7.21) yields the gap equation

$$\Omega_{i,\mathbf{k}}^2 = m_i^2 + k^2 + \frac{\lambda\hbar}{3!} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\Omega_{i,\mathbf{k}}} \coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2}). \tag{7.26}$$

The infinite quantity that appears in Eq. (7.26) will be absorbed by the bare coupling parameters m_i^2 and λ to result in the renormalized ones and a finite equation for $\Omega_{i,\mathbf{k}}^2$ [44, 12]. The solutions (7.25) hold till the quench time $t = 0$.

But after the quench time, $m^2(t)$ changes to $-m_f^2$, and therefore, the long wavelength modes become unstable and the spinodal instability begins. The long wavelength solutions are given by

$$\varphi_{f_U,\mathbf{k}}(t) = \frac{1}{\sqrt{2\hbar\Omega_{i,\mathbf{k}}}} \Big[-i\frac{\Omega_{i,\mathbf{k}}}{\tilde{\Omega}_{f,\mathbf{k}}(t)} \sinh(\int_0^t \tilde{\Omega}_{f,\mathbf{k}}(t)) + \cosh(\int_0^t \tilde{\Omega}_{f,\mathbf{k}}(t))\Big], \tag{7.27}$$

and the short wavelength solutions by

$$\varphi_{f_S,\mathbf{k}}(t) = \frac{1}{\sqrt{2\hbar\Omega_{i,\mathbf{k}}}} \Big[-i\frac{\Omega_{i,\mathbf{k}}}{\Omega_{f,\mathbf{k}}(t)} \sin(\int_0^t \Omega_{f,\mathbf{k}}(t)) + \cos(\int_0^t \Omega_{f,\mathbf{k}}(t))\Big], \tag{7.28}$$

where

$$
\begin{aligned}
\tilde{\Omega}^2_{f,\mathbf{k}}(t) &= m_f^2 - k^2 - \frac{\lambda}{3!}\phi_c^2 \\
&\quad - \frac{\lambda\hbar}{3!}\int_0^{k_\Lambda} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2})}{2\Omega_{i,\mathbf{k}}} \Big[\Big\{ \Big(\frac{\Omega_{i,\mathbf{k}}}{\tilde{\Omega}_{f,\mathbf{k}}(t)}\Big)^2 + 1 \Big\}^2 \sinh(\int_0^t \tilde{\Omega}_{f,\mathbf{k}}(t)) + 1 \Big] \\
&\quad - \frac{\lambda\hbar}{3!}\int_{k_\Lambda}^{\infty} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2})}{2\Omega_{i,\mathbf{k}}} \Big[\Big\{ \Big(\frac{\Omega_{i,\mathbf{k}}}{\Omega_{f,\mathbf{k}}(t)}\Big)^2 - 1 \Big\}^2 \sin(\int_0^t \Omega_{f,\mathbf{k}}(t)) + 1 \Big],
\end{aligned}
\tag{7.29}
$$

and $\Omega^2_{f,\mathbf{k}}(t) = -\tilde{\Omega}^2_{f,\mathbf{k}}(t)$. In contrast with the free scalar model in Sec. VI, not only the duration of the instability but also the band of unstable modes decrease in time due to the back-reaction $\hbar^2\varphi^*_{\mathbf{k}}(t)\varphi_{\mathbf{k}}(t)$ of the self-interaction and $\phi_c^2(t)$ of the classical background. The classical background obeying Eq. (7.23), though it remained around the initial vacuum under thermal fluctuations before the quench time, begins to roll down from the false vacuum to the true vacuum at first largely due to the exponentially growing unstable modes and then to the self-interaction. The competition between $-m_f^2$ and $\langle\hat{\Phi}^*_f\hat{\Phi}_f\rangle$ together ϕ_c^2 determines the spinodal line, beyond which the instability stops and the fluctuations begin to oscillate around the true vacuum. Therefore the self-interacting phase transition model has a natural exit to the spinodal instability [22].

We now turn to the case of symmetric thermal state. Once the initial condition is prescribed such that $\phi_c(-\infty) = \dot{\phi}_c(-\infty) = 0$, the classical field $\phi_c(t)$ remains in the false vacuum even during the quench and has the trivial solution $\phi_c(t) = 0$ for all times. Hence Eq. (7.23) is identically satisfied, and the dynamics of the second order phase transition is entirely governed by Eq. (7.12) for quantum fluctuations, which extends the free scalar field model considered on Sec. VI. Though each mode of quantum fluctuations has the zero expectation value, the Wigner function becomes sharply peaked around its classical trajectory as the quench proceeds [50]. Even without the classical field ϕ_c the quantum contribution from the self-interaction in Eq. (7.12) still prevents the unstable modes from growing indefinitely and provides a natural exit to the spinodal instability. This implies that the quantum dynamics of the second order phase transition is classically correlated and exhibits most of the essential points described by the order parameter under thermal fluctuations.

Finally we comment on the formation process of topological defects. The topological defects formed from the second order phase transition in this paper are domain walls. The correlation of domain walls can be determined by the two-point thermal correlation function

$$
G_{\mathrm{T}}(\mathbf{y},\mathbf{x},t) = \langle\hat{\Phi}^*(\mathbf{y},t)\hat{\Phi}(\mathbf{x},t)\rangle_{\mathrm{T}} = \int \frac{d^3k}{(2\pi)^3}\big[\hbar^2\varphi^*_{\mathbf{k}}(t)\varphi_{\mathbf{k}}(t)\coth(\frac{\beta\hbar\omega_{i,\mathbf{k}}}{2})\big]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \tag{7.30}
$$

where $\omega_{i,\mathbf{k}} = \omega_{\mathbf{k}}(t=-\infty)$. After the quench, the two-point correlation function is dominated by the unstable modes φ_{f_U}. The scaling behavior of two-point correlation function in Sec. VI holds still before reaching the spinodal line. That is, the size of domains grows according to the power law $t^{1/4}$ during the quench and the Cahn-Allen relation $t^{1/2}$ after the completion of quench. However, the domains can not grow indefinitely due to the back-reaction of the self-interaction and the classical background. As each unstable mode reaches further the spinodal line, its solution stops exponential growing and oscillates. The behavior of

the two-point thermal correlation function changes from that in Sec. VI, and one would expect a different scaling relation for the domain size, probably with a small power than the Cahn-Allen relation.

VIII. CONCLUSION

In this paper we have elaborated the recently introduced Liouville-von Neumann (LvN) approach to describe properly the time-dependent nonequilibrium systems. The systems interacting directly with environments or undergoing phase transitions are such nonequilibrium systems. These systems are characterized by time-dependent coupling parameters and their true nonequilibrium evolution deviates significantly from the equilibrium one when their coupling parameters differ greatly from their initial values. In this case the systems evolve completely out of equilibrium. For that purpose there have been developed many different methods such as the closed time-path integral method, sometimes in conjunction with the large N-expansion, mean-field, Hartree-Fock method. The LvN approach developed in this paper is a canonical method that unifies the functional Schrödinger equation for the quantum evolution of pure states and the LvN equation for the quantum description of mixed states of either equilibrium or nonequilibrium. Because the LvN approach shares all the useful techniques with quantum mechanics and quantum many-particle systems, it turns out to be a powerful method for describing time-dependent harmonic oscillators and anharmonic oscillators, and provides a rigorous and systematic method for describing time-dependent phase transitions.

By applying the LvN approach to time-dependent harmonic oscillators, we have found exactly the nonequilibrium quantum evolution evolving from various initial states such as the vacuum, number, coherent and thermal states. In this case the LvN approach is based on two operators, the so-called the annihilation and creation operators, that satisfy the quantum LvN equation, so it is straightforward to construct the Fock space of number states and the density operator according to the standard technique of quantum mechanics. We have thus obtained the density operator in terms of the classical solution, and by using the exact wave functions for number states, have been able to find the explicit form of the density matrix. In particular, the density matrix provides us with a criterion on nonequilibrium vs. equilibrium evolution. Moreover, the LvN approach has been applied to the time-dependent inverted harmonic oscillators, which can be regarded as models for second order phase transitions. For time-dependent anharmonic oscillators, we have found approximately the nonequilibrium evolution of the symmetric Gaussian, coherent and thermal states at the lowest order of the coupling constant of the quartic term. It has been shown that the LvN approach is equivalent to the effective action method and to the mean field or Hartree-Fock method.

Finally we have applied the LvN approach to the systems undergoing the symmetry breaking second order phase transition. In particular, due to the quench the coupling parameters change the sign during the evolution. As field models we have studied a free massive scalar field with an instantaneous and a finite smooth quench. By applying the LvN approach to this symmetry breaking system we have found the two-point vacuum and thermal correlation functions. It has proved that the spinodal instability leads to the $t^{1/4}$-scaling relation for domain sizes during the quench and the classical Cahn-Allen relation after the

completion of quench. The Cahn-Allen scaling relation confirms the result for the instantaneous quench model in Refs. [12,13]. One prominent feature of the finite smooth quench model is the time-lag occurring at the cubic power of the quench period in the Cahn-Allen scaling relation after the completion of quench. The inclusion of a self-interacting term shuts off the spinodal instability after crossing the spinodal line and gives rise to a natural exit for the spinodal decomposition. Not treated in detail in this paper is the very slow quench effect, which may show a transient resonance of decaying solution of long wavelength modes and will be addressed in a future research.

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APPENDIX A: HARMONIC OSCILLATOR WAVE FUNCTIONS

The vacuum state of the time-dependent oscillator is annihilated by $\hat{a}(t)$:

$$\hat{a}(t)|0,t\rangle = 0. \tag{A1}$$

In the coordinate representation

$$\Psi_0(q,t) = \langle q|0,t\rangle, \tag{A2}$$

Eq. (A1) becomes

$$i\left[u^* \frac{\hbar}{i}\frac{\partial}{\partial q} - m\dot{u}^* q\right]\Psi_0(q,t) = 0. \tag{A3}$$

We thus obtain the normalized wave function for the vacuum state

$$\Psi_0(q,t) = \left(\frac{1}{2\pi\hbar^2 u^* u}\right)^{1/4} \exp\left[\frac{i}{2}\frac{m}{\hbar}\frac{\dot{u}^*}{u^*}q^2\right]. \tag{A4}$$

The wave function for the nth number state is obtained by applying the creation operator $\hat{a}^\dagger(t)$ n-times

$$\Psi_n(q,t) = \frac{1}{\sqrt{n!}}\Big(\hat{a}^\dagger(t)\Big)^n \Psi_0(q,t). \tag{A5}$$

By making use of the relation

$$\Big(\hat{a}^\dagger(t)\Big)^n \Psi(q,t) = \Big(-\hbar u\Big)^n e^{\frac{i}{2}\frac{m}{\hbar}\frac{\dot{u}}{u}q^2}\Big(\frac{\partial}{\partial q}\Big)^n \left(e^{-\frac{i}{2}\frac{m}{\hbar}\frac{\dot{u}}{u}q^2}\Psi(q,t)\right), \tag{A6}$$

and the definition of the Hermite polynomial

$$H_n(x) = (-1)^n e^{x^2}\Big(\frac{d}{dx}\Big)^n e^{-x^2}, \tag{A7}$$

we obtain the wave function

$$\Psi_n(q,t) = \left(\frac{1}{2\pi\hbar^2 u^* u}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \left(\frac{u}{u^*}\right)^n H_n(x) \exp\left[\frac{i}{2}\frac{m}{\hbar}\frac{\dot{u}^*}{u^*} q^2\right], \tag{A8}$$

where

$$x = \frac{q}{\sqrt{2\hbar^2 u^* u}}. \tag{A9}$$

Here we have also used the wronskian (3.5).

APPENDIX B: DENSITY MATRIX

In the coordinate representation, the density operator defined by

$$\hat{\rho}_{\mathrm{T}}(t) = \frac{1}{Z_N} e^{-\beta\hbar\omega_0(\hat{N}(t)+\frac{1}{2})}, \tag{B1}$$

where Z_N is the partition function, becomes

$$\rho_{\mathrm{T}}(q', q, t) = \frac{1}{Z_N} \sum_{n=0}^{\infty} \Psi_n(q', t)\Psi_n^*(q, t) e^{-\beta\hbar\omega_0(n+\frac{1}{2})}. \tag{B2}$$

By substituting (A8) into (B2), we obtain

$$\rho_{\mathrm{T}}(q', q, t) = \frac{1}{Z_N}\left(\frac{1}{2\pi\hbar^2 u^* u}\right)^{1/2} \sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n(x') H_n(x) e^{-\beta\hbar\omega_0(n+\frac{1}{2})} e^{\frac{i}{2}\frac{m}{\hbar}\frac{\dot{u}^*}{u^*}q'^2 - \frac{i}{2}\frac{m}{\hbar}\frac{\dot{u}}{u}q^2}. \tag{B3}$$

Following Kubo's method [40], we rewrite the product of Hermite polynomials as [43]

$$H_n(x')H_n(x) = \frac{1}{\pi} e^{x'^2+x^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dz_1 dz_2 (2iz_1)^n (2iz_2)^n e^{-z_1^2 - 2ix'z_1 - z_2^2 - 2ixz_2} \tag{B4}$$

and sum over n to obtain

$$\rho_{\mathrm{T}}(q', q, t) = \frac{1}{Z_N}\left(\frac{1}{2\pi\hbar^2 u^* u}\right)^{1/2} \frac{e^{-\frac{\beta\hbar\omega_0}{2}}}{\pi} \exp\left[(x'^2 + x^2) + \frac{i}{2}\frac{m}{\hbar}\left\{\frac{\dot{u}^*}{u^*}q'^2 - \frac{\dot{u}}{u}q^2\right\}\right]$$
$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dz_1 dz_2 \exp\left[-z_1^2 - 2ix'z_1 - z_2^2 - 2ixz_2 - 2z_1 z_2 e^{-\beta\hbar\omega_0}\right]. \tag{B5}$$

After doing the integral and using the identity

$$x^2 = \frac{i}{2}\frac{m}{\hbar}\left(\frac{\dot{u}}{u} - \frac{\dot{u}^*}{u^*}\right) q^2, \tag{B6}$$

we finally obtain

$$\rho_{\mathrm{T}}(q', q, t) = \frac{1}{Z_N}\left[\frac{1}{4\pi\hbar^2 u^* u \sinh(\beta\hbar\omega_0)}\right]^{1/2} \exp\left[\frac{i}{2}\hbar m \frac{d}{dt}\ln(u^* u)(x'^2 - x^2)\right]$$
$$\times \exp\left[-\frac{1}{4}\left\{(x'+x)^2 \tanh\left(\frac{\beta\hbar\omega_0}{2}\right) + (x'-x)^2 \coth\left(\frac{\beta\hbar\omega_0}{2}\right)\right\}\right]. \tag{B7}$$

In the special case of the time-independent oscillator we recover the classical result by Kubo [40] by substituting the complex solution

$$u(t) = \frac{e^{-i\omega_0 t}}{\sqrt{2\hbar m\omega_0}} \tag{B8}$$

into Eq. (B7).

APPENDIX C: STEEPEST DECENT METHOD

The integral appearing in the two-point correlation function has the form

$$I = \int dx x e^{-\gamma x^2} \sin(yx) F(x) \tag{C1}$$

where $F(x)$ is a slowly varying function. Note that $xe^{-\gamma x^2}$ is a highly peaked function that varies rapidly. We let

$$e^{g(x)} \equiv xe^{-\gamma x^2} \tag{C2}$$

where

$$g(x) = -\gamma x^2 + \ln(x). \tag{C3}$$

We expand $g(x)$ in a Taylor series and truncate it up to the quadratic term around the maximum point $x_0 = \frac{1}{\sqrt{2\gamma}}$

$$g(x) \approx g(x_0) - 2\gamma(x - x_0)^2, \tag{C4}$$

and rewrite the integrand as

$$\begin{aligned} xe^{-\gamma x^2} \sin(yx) F(x) &\approx x_0 e^{-\gamma x_0^2} F(x_0) e^{-2\gamma(x-x_0)^2} \frac{e^{iyx} - e^{-iyx}}{2i} \\ &= x_0 e^{-\gamma x_0^2} F(x_0) \Bigg\{ \exp\Big[-2\gamma\Big(x - x_0 - i\frac{y}{4\gamma}\Big)^2\Big] \frac{e^{iyx_0}}{2i} \\ &\quad + \exp\Big[-2\gamma\Big(x - x_0 + i\frac{y}{4\gamma}\Big)^2\Big] \frac{(-1)e^{-iyx_0}}{2i} \Bigg\} e^{-\frac{y^2}{8\gamma}}. \end{aligned} \tag{C5}$$

The Gaussian integrals contribute equally, so we obtain

$$I \approx \left(\frac{\pi}{4\gamma}\right)^{1/2} x_0 e^{-\gamma x_0^2} F(x_0) \sin(yx_0) e^{-\frac{y^2}{8\gamma}}. \tag{C6}$$

APPENDIX D: ASYMPTOTIC FORM FOR UNSTABLE MODES IN A FINITE QUENCH

To find the contribution to the two-point functions from the unstable growing modes for a finite quench ($\tilde{\omega}_{f,\mathbf{k}}\tau < 1$), we need to evaluate $\Gamma(\tilde{\omega}_{f,\mathbf{k}})$ and $\Gamma(\frac{\tau}{2}(\tilde{\omega}_{f,\mathbf{k}} - i\omega_{i,\mathbf{k}}))$. These gamma functions are rewritten as

$$\begin{aligned}\Gamma(\tilde{\omega}_{f,\mathbf{k}}\tau) &= \frac{\Gamma(1+\tilde{\omega}_{f,\mathbf{k}}\tau)}{\tilde{\omega}_{f,\mathbf{k}}\tau},\\ \Gamma(\frac{\tau}{2}(\tilde{\omega}_{f,\mathbf{k}} - i\omega_{i,\mathbf{k}})) &= \frac{\Gamma(1+\frac{\tau}{2}(\tilde{\omega}_{f,\mathbf{k}} - i\omega_{i,\mathbf{k}}))}{\frac{\tau}{2}(\tilde{\omega}_{f,\mathbf{k}} - i\omega_{i,\mathbf{k}})}.\end{aligned} \tag{D1}$$

We further make use of the expansion formula [42]

$$\ln\Gamma(1+z) = -\ln(1+z) + (1-\gamma)z + \sum_{n=2}^{\infty}(-1)^n\Big[\zeta(n)-1\Big]\frac{z^n}{n}, \quad (|z|<2), \tag{D2}$$

where

$$\gamma = \lim_{m\to\infty}\left[\sum_{k=1}^{m}\frac{1}{k} - \ln(m)\right] = 0.5772156649\cdots \tag{D3}$$

is the Euler's constant and $\zeta(n)$ is the Riemann Zeta function

$$\zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n}. \tag{D4}$$

Here

$$z = \tilde{\omega}_{f,\mathbf{k}}\tau, \quad z = \frac{\tau}{2}(\tilde{\omega}_{f,\mathbf{k}} - i\omega_{i,\mathbf{k}}\tau). \tag{D5}$$

We now expand the gamma functions up to the cubic power of τ:

$$\begin{aligned}\Gamma(1+\tilde{\omega}_{f,\mathbf{k}}\tau) = \frac{1}{1+\tilde{\omega}_{f,\mathbf{k}}\tau}\exp\Big[&(1-\gamma)(\tilde{\omega}_{f,\mathbf{k}}\tau) + \frac{1}{2}[\zeta(2)-1](\tilde{\omega}_{f,\mathbf{k}}\tau)^2\\ &-\frac{1}{3}[\zeta(3)-1](\tilde{\omega}_{f,\mathbf{k}}\tau)^3 + \mathcal{O}(\tau^4)\Big]\end{aligned} \tag{D6}$$

and

$$\begin{aligned}\Big|\Gamma(1+\frac{\tau}{2}(\tilde{\omega}_{f,\mathbf{k}} - i\omega_{i,\mathbf{k}}))\Big|^2 = \frac{1}{|1+\frac{\tau}{2}(\tilde{\omega}_{f,\mathbf{k}} - i\omega_{i,\mathbf{k}})|^2}\exp\Big[&(1-\gamma)(\tilde{\omega}_{f,\mathbf{k}}\tau)\\ &+\frac{1}{2}[\zeta(2)-1](\frac{\tau}{2})^2\Big\{(\tilde{\omega}_{f,\mathbf{k}} - i\omega_{i,\mathbf{k}})^2 + (\tilde{\omega}_{f,\mathbf{k}} + i\omega_{i,\mathbf{k}})^2\Big\}\\ &-\frac{1}{3}[\zeta(3)-1](\frac{\tau}{2})^3\Big\{(\tilde{\omega}_{f,\mathbf{k}} - i\omega_{i,\mathbf{k}})^3 + (\tilde{\omega}_{f,\mathbf{k}} - i\omega_{i,\mathbf{k}})^3\Big\} + \mathcal{O}(\tau^4)\Big].\end{aligned} \tag{D7}$$

Therefore it follows that

$$\left|\frac{\Gamma(1+\tilde{\omega}_{f,\mathbf{k}}\tau)}{\Gamma^2(1+\frac{\tau}{2}(\tilde{\omega}_{f,\mathbf{k}}-i\omega_{i,\mathbf{k}}))}\right|^2 = \left[\frac{\left(1+\frac{\tau}{2}\tilde{\omega}_{f,\mathbf{k}}\right)^2+\frac{\tau^2}{4}\omega_{i,\mathbf{k}}^2}{1+\tilde{\omega}_{f,\mathbf{k}}\tau}\right]^2$$
$$\times\exp\left[\frac{1}{2}[\zeta(2)-1]\tau^2\left(\tilde{\omega}_{f,\mathbf{k}}^2+\omega_{i,\mathbf{k}}^2\right)-\frac{1}{4}[\zeta(3)-1]\tau^3\tilde{\omega}_{f,\mathbf{k}}\left(\tilde{\omega}_{f,\mathbf{k}}^2+\omega_{i,\mathbf{k}}^2\right)+\mathcal{O}(\tau^4)\right]. \tag{D8}$$

We finally get

$$\left|\frac{\Gamma(\tilde{\omega}_{f,\mathbf{k}}\tau)}{\Gamma^2(\frac{\tau}{2}(\tilde{\omega}_{f,\mathbf{k}}-i\omega_{i,\mathbf{k}}))}\right|^2 = \left[\frac{\frac{\tau^2}{4}\left(\tilde{\omega}_{f,\mathbf{k}}^2+\omega_{i,\mathbf{k}}^2\right)}{\tilde{\omega}_{f,\mathbf{k}}\tau}\right]^2\left[\frac{\left(1+\frac{\tau}{2}\tilde{\omega}_{f,\mathbf{k}}\right)^2+\frac{\tau^2}{4}\omega_{i,\mathbf{k}}^2}{1+\tilde{\omega}_{f,\mathbf{k}}\tau}\right]^2$$
$$\times\exp\left[\frac{1}{2}[\zeta(2)-1]\tau^2\left(\tilde{\omega}_{f,\mathbf{k}}^2+\omega_{i,\mathbf{k}}^2\right)-\frac{1}{4}[\zeta(3)-1]\tau^3\tilde{\omega}_{f,\mathbf{k}}\left(\tilde{\omega}_{f,\mathbf{k}}^2+\omega_{i,\mathbf{k}}^2\right)+\mathcal{O}(\tau^4)\right]. \tag{D9}$$

The last relation necessary for the two-point correlation function is [42]

$$\left|\Gamma(1-i\omega_{i,\mathbf{k}}\tau)\right|^2 = \frac{\pi\omega_{i,\mathbf{k}}\tau}{\sinh(\pi\omega_{i,\mathbf{k}}\tau)}. \tag{D10}$$

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